XII. *On* STOKES'S *Current Function.*

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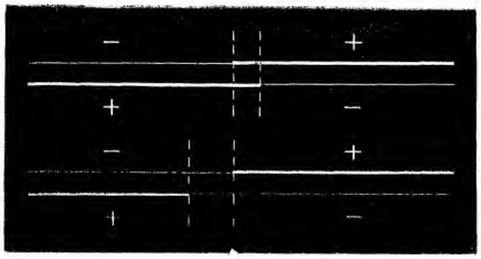
IN MAXWELL'S 'Electricity and Magnetism'[*] a view is put forward, in accordance with which we may regard any irrotational motion in a perfect liquid, for which the velocity potential is a solid zonal harmonic, as due to the juxtaposition at the origin, and upon the axis of symmetry, of sinks and sources.

But, in a liquid, any irrotational motion which is symmetrical with respect to an axis gives a velocity potential which may be expressed as a sum of a series of solid zonal harmonics, their common axis being the axis of symmetry, and their origin arbitrary, provided it is excluded from the region to which the expressions apply. The position of the origin upon the axis is arbitrary, since, by a transference formula, we may pass from one origin to another.

Let us now consider the system formed by a line source and a line sink, of equal strengths, extending along the axis from an arbitrary origin to infinity in opposite directions. Such a system I shall call an *extended doublet*, of strength m, where m is the strength per unit length of that part which lies on the positive side of the origin.



By the superposition of two extended doublets of equal but opposite strengths we can produce a sink or a source upon the axis. Hence, in a liquid, any irrotational motion which is symmetrical with respect to an axis may be produced by superposition of extended doublets, whose origins depart but little from an arbitrary point on the axis of symmetry.

Now, for an extended doublet of strength m, STOKES'S Current Function ψ, for any point distant r from the origin, is $-2mr$. For let ζ be the distance of the origin of the doublet from the origin of coordinates, and let $\psi(m, \zeta)$ be the value of STOKES's Current Function for any point (ϖ, z).

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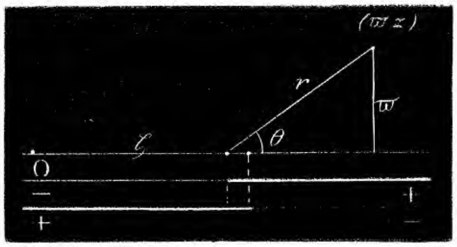
[*] Vol. 1, chapter ix.



Then if $\delta\psi$ be the Current Function for a source of strength $2m\ \delta\zeta$ at the point ζ of the axis we get

$$\frac{1}{\varpi}\frac{d}{dr}\delta\psi = 0,$$

$$\frac{1}{\varpi}\frac{d}{rd\theta}\delta\psi = \frac{2m\delta\zeta}{r^2}.$$

Therefore

$$\frac{d}{d\theta}\delta\psi = 2m\frac{\varpi}{r}\delta\zeta,$$

$$= 2m\sin\theta\ \delta\zeta,$$

whence

$$\delta\psi = -2m\delta\zeta\cos\theta,$$

the constant of integration being zero.

But

$$\delta\psi = \psi(m, \zeta) + \psi(-m, \zeta + \delta\zeta),$$

and clearly,

$$\psi(-m, \zeta + \delta\zeta) = -\psi(m, \zeta + \delta\zeta).$$

Hence

$$\psi(m, \zeta) - \left[\psi(m, \zeta) + \frac{d\psi}{d\zeta}\delta\zeta\right],$$

$$= -\frac{d\psi(m, \zeta)}{d\zeta}\delta\zeta,$$

$$= -2m\delta\zeta\cos\theta,$$

$$= -2m\delta\zeta\frac{z-\zeta}{\sqrt{\{\varpi^2 + (z-\zeta)^2\}}}.$$

Therefore

$$\frac{d\psi}{d\zeta} = 2m\frac{z-\zeta}{\sqrt{\{\varpi^2 + (z-\zeta)^2\}}},$$

and

$$\psi = -2mr \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where

$$r = \sqrt{\{\varpi^2 + (z - \zeta)^2\}},$$

disregarding a constant.

Thus, if

$$m = f(\zeta)\, d\zeta,$$

we may, by properly choosing the function f, write

$$\psi = \int_{-\infty}^{+\infty} f(\zeta) \sqrt{\{\varpi^2 + (z - \zeta)^2\}}\, d\zeta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2),$$

where ψ is the current function for any irrotational motion in a liquid, symmetrical about the axis of z.

Again, if

$$r = \sqrt{\{\varpi^2 + (z - \zeta)^2\}},$$

$$\frac{dr}{d\varpi} = \frac{\varpi}{r}, \qquad \frac{dr}{dz} = \frac{z - \zeta}{r},$$

$$\frac{d^2r}{d\varpi^2} = \frac{1}{r} - \frac{\varpi^2}{r^3}, \qquad \frac{d^2r}{dz^2} = \frac{1}{r} - \frac{(z - \zeta)^2}{r^3}.$$

Therefore

$$\begin{aligned} \frac{d^2r}{d\varpi^2} + \frac{d^2r}{dz^2} &= \frac{2}{r} - \frac{\varpi^2 + (z - \zeta)^2}{r^3}, \\ &= \frac{1}{r}, \\ &= \frac{1}{\varpi}\frac{dr}{d\varpi}, \end{aligned}$$

and the expression (1), and consequently also (2) satisfies the differential equation

$$\frac{d^2\psi}{d\varpi^2} + \frac{d^2\psi}{dz^2} - \frac{1}{\varpi}\frac{d\psi}{d\varpi} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3),$$

or, as I shall write it, $D\psi = 0$.

When the motion is rotational, (3) no longer holds. In fact, as is well known, we have under all circumstances

$$\frac{1}{\varpi} D\psi = -2\omega,$$

where ω is the resultant spin at the point (ϖ, z).

Thus, if there is spin in the fluid, (3) is replaced by

$$\frac{d^2\psi}{d\varpi^2} + \frac{d^2\psi}{dz^2} - \frac{1}{\varpi}\frac{d\psi}{d\varpi} = -2\varpi\omega \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3a).$$

Again, if ∇^2 stand for the operator

$$\frac{d^2}{d\varpi^2} + \frac{d^2}{dz^2} + \frac{1}{\varpi}\frac{d}{d\varpi} + \frac{1}{\varpi^2}\frac{d^2}{d\phi^2},$$

ϕ being the azimuthal angle about the axis of symmetry, it may be seen at once that

$$\text{D}\psi = \frac{\varpi}{\sin\phi}\nabla^2\frac{\sin\phi}{\varpi}\psi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

Consequently, $(3a)$ may be written

$$\nabla^2\frac{\psi\sin\phi}{\varpi} = -2\omega\sin\phi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3a').$$

Consequently,

$$\psi = \psi_o + \frac{\varpi}{2\pi\sin\phi}\iiint\frac{\sin\phi'\omega'\,dx'\,dy'\,dz'}{r},$$

where ψ_o is a solution of (3), or, ψ consists of a solution of (3), together with $\frac{\varpi}{2\pi\sin\phi}$ × the potential at the point considered of a distribution of mass of density at any point, $\sin\phi$ × the spin at that point. This result is given by BASSET, 'Hydrodynamics,' vol. 2, § 306.

I give one other general result. Since

$$\omega = -\frac{1}{2\varpi}\text{D}\psi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3a),$$

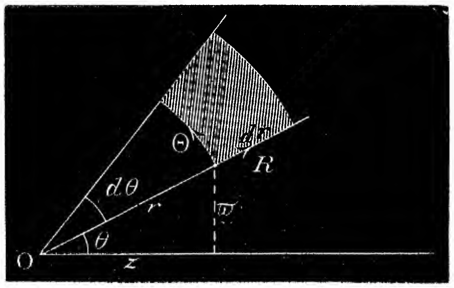
the circulation in any evanescible circuit drawn in a meridional plane is

$$-\iint\frac{1}{\varpi}\text{D}\psi\,d\varpi\,dz \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where the integration extends over the area embraced by the circuit.

This result enables us to transform $\text{D}\psi$ readily from cylindrical to other systems of coordinates. For instance, consider polar coordinates, r, θ, and let us find the circulation in a small rectangle bounded by r, $r + dr$, θ, $\theta + d\theta$.

Let the velocities in the direction of r, and perpendicular to it, be R, Θ.



Then the circulation in this circuit is

$$\mathrm{R}\,dr + \left[\Theta r + \frac{d}{dr}(\Theta r)\,dr\right]d\theta - \left[\mathrm{R} + \frac{d\mathrm{R}}{d\theta}\,d\theta\right]dr - \Theta r\,d\theta$$

$$= r\,dr\,d\theta\left[\frac{d\Theta}{dr} + \frac{\Theta}{r} - \frac{d\mathrm{R}}{rd\theta}\right].$$

Now

$$\Theta = -\frac{1}{r\sin\theta}\frac{d\psi}{dr}$$

$$\mathrm{R} = \frac{1}{r^2\sin\theta}\frac{d\psi}{d\theta}.$$

Thus the expression in square brackets is

$$-\frac{1}{r\sin\theta}\left[\frac{d^2\psi}{dr^2} + \frac{1}{r^2}\left(\frac{d^2\psi}{d\theta^2} - \cot\theta\frac{d\psi}{d\theta}\right)\right],$$

or,

$$\mathrm{D}\psi = \frac{d^2\psi}{dr^2} + \frac{\sin\theta}{r^2}\frac{d}{d\theta}\left(\frac{1}{\sin\theta}\frac{d\psi}{d\theta}\right)$$

$$= \frac{d^2\psi}{dr^2} + \frac{1-\mu^2}{r}\frac{d^2\psi}{d\mu^2} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (6),$$

if μ stands for $\cos\theta$.

Other applications will be found later.

Reverting now to the expression (1), it will be seen that the direct distance of any point from a point on the axis of symmetry, plays the same part in the theory of STOKES's Current Function that is played by its reciprocal in the theory of the potential function belonging to symmetrical distributions of matter.

Thus, if r_o, 0, r, θ be the coordinates of a point upon the axis, and of any other point, the distance between these points, $\sqrt{(r_o^2 - 2r_o r\cos\theta + r^2)}$, may be developed in a converged series, say

$$\sum_{n=0}^{n=\infty} -\frac{r^n}{r_o^{n-1}}\mathrm{I}_n(\cos\theta) \qquad \text{or} \qquad \sum_{n=0}^{n=\infty} -\frac{r_o^n}{r^{n-1}}\mathrm{I}_n(\cos\theta)$$

according as r_o is greater or less than r, $\mathrm{I}_n(\cos\theta)$ being a certain function of θ, and we see from (6) that

$$(1-\mu^2)\frac{d^2\mathrm{I}_n(\mu)}{d\mu^2} + n(n-1)\,\mathrm{I}_n(\mu) = 0 \quad . \; . \; . \; . \; . \; . \; . \quad (7).$$

Now it is evident from the analogue of Zonal Harmonics, that it is proper to discuss the function $\mathrm{I}_n(\cos\theta)$, and other solutions of (7) before considering the applications of STOKES's Current Function to the motion of liquids. It is with this discussion that the first three chapters are occupied, and, as might be expected, the theory closely resembles that of Spherical Harmonics. I have accordingly made free use of the order and methods adopted by HEINE, in his 'Handbuch d. Kugelfunctionen,' more especially in Chapters I. and II., where the necessary changes were slight. Moreover, the functions I deal with have themselves been discussed by HEINE, on a different method, and most of the expressions which I find in the following pages are given by him. Full references to these are given on p. 461.

The idea of developing the solutions of $D\psi = 0$ in a manner more or less analogous to that employed with regard to LAPLACE's equation, appears to have been first used by O. E. MEYER,* who obtains the equation (7), shows that the functions contain $1 - \mu^2$ as a factor, and that they obey (22A), Chapter II. An expression which shows the relation of the functions to Zonal Harmonics was given by Mr. BUTCHER,† and functions of fractional order have been used by Mr. HICKS,‡ in connection with his researches on the theory of the motion of vortex rings. The fuller account of such functions, which is found in the following pages, may be of interest in relation to these; for example, I would refer to p. 44.

The applications to Hydrodynamics, which I here give, are of mathematical interest rather than physical. They are chiefly in connection with the motion of viscous liquids. In 'Crelle-Borchardt,' vol. 81, OBERBECK has given the velocities produced in an infinite viscous liquid by the steady motion of an ellipsoid through it, in the direction of one of its axes, and from these Mr. HERMAN§ has found the equation of a family of surfaces containing the stream-lines relative to the ellipsoid. In Chapter VI., STOKES's current function is obtained by a direct process for the flux of a viscous liquid past a spheroid, and it is shown that the result differs only by a constant multiple from the particular case of Mr. HERMAN's integral.

Some minor applications are also given; namely, the solutions are obtained for flux past an approximate sphere, and past an approximate spheroid. The solution is also obtained for flux through a hyperboloid of one sheet, where it appears that the stream surfaces are hyperboloids of the confocal system. A particular case is that of flux through a circular hole in a wall, and this is interesting, because we see that by supposing internal friction to take place in the liquid, we find an expression which gives zero velocity at the sharp edge, and thus avoids the difficulty which is always present in the solution of such problems, on the supposition that the liquid is perfect. A comparison may be instituted between this problem and that of the effect of a disturbing periodic force upon a dynamical system capable of vibrating alone with a period equal to that of the force. It is well known that the amplitude of the vibration induced appears infinite, if we totally disregard friction, and this difficulty is met by the fact that the damping effect of even slight friction is increased considerably by high velocities. Now a viscous liquid can move irrotationally, and if there were no friction at the boundaries this is the class of motion it would take in cases of flux past or through obstacles. But if the obstacle terminated in a sharp edge, this would make the velocity there infinite, and the friction, however inconsiderable elsewhere, would here become of account. The boundary conditions which were necessary for the existence of irrotational motion throughout the liquid would no longer apply, and the whole character of the solution would be changed.

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* 'CRELLE,' vol. 73.

† 'London Math. Soc. Proc.,' vol. 8, see p. 49.

‡ 'Phil. Trans.,' 1884; 1885.

§ 'Quart. Journ. Math.,' 1889 (No. 92).

This would, at any rate, seem to apply to cases in which the whole motion is slow, and when, consequently, the boundary conditions which must hold are pretty well understood.

The paper concludes with an attempt to discuss the flux past a spheroid or through a hyperboloid at whose boundary there may be slipping. The current function is not obtained, all that appears being that it probably differs from the parallel case of the sphere in being far more complicated than when there is no slipping. From this we except the case of flux through a circular hole in a plane wall, when the solution for no slipping satisfies the new conditions.

CHAPTER I.—VARIOUS FORMS OF $I_n(x)$.

WHERE the method of development in the following pages does not necessarily differ from that of HEINE, I confine myself to a statement of result, with an indication of the method, and a reference to the corresponding section in his 'Handbuch,' vol. 1. In one place the reference is to FERRERS's 'Spherical Harmonics.'

Let R denote $\sqrt{(1 - 2\alpha x + x^2)}$.

Then writing

$$R = -\Sigma\alpha^n I_n(r) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

we find

$$I_n(r) = \frac{1.3\ldots 2n-3}{1.2\ldots n}\left[x^n - \frac{n(n-1)}{2(2n-3)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4.(2n-3)(2n-5)}x^{n-4}\ldots\right]$$

the series ending with x^1 or x^0,

$$\left.\begin{aligned} &= 2^{n-1}\frac{\Pi(n-\frac{3}{2})}{\Pi(n)\,\Pi(-\frac{1}{2})}x^n\,F\left(-\frac{n}{2}, -\frac{n}{2}+\tfrac{1}{2}, -n+\tfrac{3}{2}, x^{-2}\right) \\ &= 2^{n-1}\frac{\Pi(n-\frac{3}{2})}{\Pi(n)\,\Pi(-\frac{1}{2})}x^{n-2}(x^2-1)\,F\left(\frac{n}{2}+\tfrac{3}{2}, 1-\frac{n}{2}, \tfrac{3}{2}-n, x^{-2}\right) \end{aligned}\right\} \quad . \quad . \quad (9)$$

n even]

$$\left.\begin{aligned} &= (-1)^{n/2}\frac{\Pi\left(\frac{n}{2}-\frac{3}{2}\right)}{2\Pi\left(\frac{n}{2}\right)\Pi\left(-\frac{1}{2}\right)}\,\mathrm{F}\left(-\frac{n}{2},\frac{n}{2}-\frac{1}{2},\frac{1}{2},x^2\right) \\ &= (-1)^{n/2}\frac{\Pi\left(\frac{n}{2}-\frac{3}{2}\right)}{2\Pi\left(\frac{n}{2}\right)\Pi\left(-\frac{1}{2}\right)}(1-x^2)\,\mathrm{F}\left(\frac{n}{2}+\frac{1}{2},-\frac{n}{2}+1,\frac{1}{2},x^2\right) \end{aligned}\right\}\quad . \quad . \quad (9\text{A})$$

n odd]

$$\left.\begin{aligned} &= (-1)^{(n-1)/2}\frac{\Pi\left(\frac{n}{2}-1\right)}{\Pi\left(\frac{n}{2}-\frac{1}{2}\right)\Pi\left(-\frac{1}{2}\right)}\,x\mathrm{F}\left(-\frac{n}{2}+\frac{1}{2},\frac{n}{2},\frac{3}{2},x^2\right) \\ &= (-1)^{(n-1)/2}\frac{\Pi\left(\frac{n}{2}-1\right)}{\Pi\left(\frac{n}{2}-\frac{1}{2}\right)\Pi\left(-\frac{1}{2}\right)}\,x(1-x^2)\,\mathrm{F}\left(\frac{n}{2}+1,-\frac{n}{2}+\frac{3}{2},\frac{3}{2},x^2\right) \end{aligned}\right\}\quad . \quad . \quad (9\text{B})$$

There are two exceptional cases, viz.,

$$\mathrm{I}_1(x) = x\,; \qquad \mathrm{I}_0(x) = -1.$$

The first eleven functions are given in Chapter IV., Table II. [*Cf.* HEINE, § 4.]

If

$$x = \cos\theta, \quad \text{and} \quad \xi = x - \sqrt{(x^2-1)} = e^{i\theta},$$

and therefore

$$2\cos n\theta = \xi^n + \xi^{-n},$$

such a sign being given to the square root as shall make mod. $\xi \not> 1$, and therefore such a sign to $i\theta$ as shall make its real part, if not zero, negative, we have

$$\left.\begin{aligned} \mathrm{I}_n(x) &= \frac{\Pi\left(n-\frac{3}{2}\right)}{2\Pi(n)\Pi\left(-\frac{1}{2}\right)}\,\xi^{-n}\,\mathrm{F}\left(-\frac{1}{2},-n,-n+\frac{3}{2},\xi^2\right) \\ \mathrm{I}_n(\cos\theta) &= \frac{\Pi\left(n-\frac{3}{2}\right)}{2\Pi(n)\Pi\left(-\frac{1}{2}\right)}\,\xi^{-n}(1-\xi^2)^2\,\mathrm{F}\left(-n+2,\frac{3}{2},-n+\frac{3}{2},\xi^2\right) \end{aligned}\right\}\quad (10)$$

$$\begin{aligned} = \frac{\Pi\left(n-\frac{3}{2}\right)}{\Pi(n)\Pi\left(-\frac{1}{2}\right)}\Bigg[& 2\cos n\theta - \frac{2n}{2(2n-3)}\,.\,2\cos(n-2)\theta \\ & - \frac{2n(2n-2)\,.\,1}{2\,.\,4\,.\,(2n-3)(2n-5)}\,.\,2\cos(n-4)\theta \\ & - \frac{2n(2n-2)(2n-4)\,.\,1\,.\,3}{2\,.\,4\,.\,6\,(2n-3)(2n-5)(2n-1)}\,.\,2\cos(n-6)\theta\,.\,.\,.\Bigg] \quad (10\text{A}) \end{aligned}$$

n being in each case a positive integer. [HEINE, § 5.]

We shall show (p. 471)

$$I_n(x) = \frac{\xi^{-n}}{\pi}\int_{\xi^2}^{1} u^{n-2}(1-u)^{\frac{1}{2}}(u-\xi^2)^{\frac{1}{2}}\,du \quad . \quad . \quad . \quad . \quad . \quad (11).$$

This form is equivalent to (10) when n is a positive integer.

Putting

$$\alpha y = 1 - \sqrt{(1 - 2\alpha x + \alpha^2)},$$

so that

$$y = x + \frac{\alpha}{2}(y^2 - 1);$$

whence, by the help of LAGRANGE'S theorem,

$$I_n(x) = \frac{1}{\underline{|n-1}}\left(\frac{d}{dx}\right)^{n-2}\left(\frac{x^2-1}{2}\right)^{n-1} \cdot \quad \text{[HEINE, § 7]} \quad . \quad (12).$$

The distance between two points whose coordinates are ϖ, z; o, ζ; is

$$\begin{aligned}&\sqrt{\{\varpi^2 + (z-\zeta)^2\}}\\ &= r\sqrt{(1 - 2xh + h^2)},\end{aligned}$$

where

$$x = z/r, \qquad h = \zeta/r, \qquad r = \sqrt{(\varpi^2 + z^2)};$$

whence, by TAYLOR'S theorem,

$$I_n(x) = (-1)^{n-1}\frac{r^{n-1}}{\underline{|n}}\cdot\frac{d^n r}{dz^n} \quad \text{[FERRERS, § 5]} \quad . \quad . \quad (13).$$

HEINE shews, §§ 8, 9, that

$$\int_0^{\pi}\frac{d\eta}{\alpha x - 1 - \alpha\cos\eta\sqrt{(x^2-1)}} = -\frac{\pi}{\sqrt{(1 - 2\alpha x + \alpha^2)}},$$

provided mod. $\alpha\sqrt{x^2-1} <$ mod. $(\alpha x - 1)$, a result which is secured by taking α small enough.

Hence

$$\begin{aligned}&1 - \alpha x - \sqrt{(1 - 2\alpha x + \alpha^2)}\\ &= \frac{1}{\pi}\int_0^{\pi} d\eta\left[1 - \alpha x + \frac{1 - 2\alpha x + \alpha^2}{\alpha x - 1 - \alpha\sqrt{x^2-1}\,.\cos\eta}\right]\\ &= \frac{1}{\pi}\int_0^{\pi} d\eta\,.\,\frac{\alpha^2(1 - x^2 + x\sqrt{x^2-1}\,.\cos\eta) - \alpha\cos\eta\sqrt{x^2-1}}{\alpha(x - \cos\eta\sqrt{x^2-1}) - 1}.\end{aligned}$$

Let each expression be developed in ascending powers of α, and equate coefficients of α^n.

Then

$$\mathrm{I}_n(x) = \frac{1}{\pi}\int_0^{\pi} (x^2 - 1) \sin^2 \eta \, (x - \cos \eta \sqrt{x^2 - 1})^{n-2} \, d\eta \quad . \quad . \quad . \quad (14),$$

$$= \frac{1}{\pi}\int_0^{\pi} (x^2 - 1) \sin^2 \phi \, (x + \cos \phi \sqrt{x^2 - 1})^{-n-1} \, d\phi \quad . \quad . \quad (14\mathrm{A}),$$

where

$$1 = (x - \cos \eta \sqrt{x^2 - 1})(x + \cos \phi \sqrt{x^2 - 1}) \quad [\text{Heine}, \S\, 9].$$

The form (14A) is inadmissible when x is a pure imaginary; for suppose the path of integration of η to include real values only; then ϕ is complex and discontinuous, passing at a step from $0 + i\infty$ to $\pi + i\infty$.

The forms (14) and (14A) may be derived from (11) by substituting

$$u/\xi = x - \cos \eta \sqrt{x^2 - 1}.$$

From (14) we find a form analogous to one of MEHLER's forms for the Zonal Harmonic.

Write

$$x - \cos \eta \sqrt{(x^2 - 1)} = e^{\theta},$$

where θ will in general be complex, and we shall suppose its imaginary part to lie between $\pm i\pi$.

Then

$$\sin \eta \sqrt{(x^2 - 1)} = \sqrt{(-1 + 2xe^{\theta} - e^{2\theta})}$$

if such a sign is given to the square root at the right as shall make its real part of like sign into $\sqrt{(x^2 - 1)}$.

Thus

$$\mathrm{I}_n(x) = \frac{1}{\pi}\int_{\theta_0}^{\theta_1} d\theta \sqrt{\{2(x - \cosh \theta)\}} \, e^{(n-\frac{1}{2})\theta} \quad . \quad . \quad . \quad . \quad . \quad . \quad (15),$$

where

$$x - \sqrt{(x^2 - 1)} = e^{\theta_0}$$
$$x + \sqrt{(x^2 - 1)} = e^{\theta_1}.$$

Thus if $x = \cosh \vartheta$, and the imaginary part of ϑ lie between $\pm i\pi$,

$$\theta_0 = -\vartheta, \qquad \theta_1 = +\vartheta,$$

and

$$\mathrm{I}_n(x) = \frac{1}{\pi}\int_{-\vartheta}^{\vartheta} d\theta \sqrt{\{2(x - \cosh \theta)\}} \, [\cosh (n - \tfrac{1}{2})\theta + \sinh (n - \tfrac{1}{2})\theta]$$

$$= -\frac{2}{\pi}\int_0^{\cosh^{-1} x} d\theta \sqrt{\{2(x - \cosh \theta)\}} \cosh (n - \tfrac{1}{2})\theta \quad . \quad . \quad . \quad . \quad (15\mathrm{A}),$$

where $\sinh(\cosh^{-1} x) = -\sqrt{(x^2-1)}$, and the imaginary part of $\cosh^{-1} x$ lies between $\pm i\pi$.

Write $i\phi$ for θ

$$\mathrm{I}_n(x) = -\frac{2}{\pi}\int_0^{\cos^{-1}x} d\phi \sqrt{\{2(\cos\phi - x)\}} \cos(n-\tfrac{1}{2})\phi \quad . \quad . \quad . \quad (15\text{B}),$$

where $\sin(\cos^{-1} x) = \sqrt{(1-x^2)}$, and the real part of $\cos^{-1} x$ lies between $\pm\pi$.

Another form may be obtained when x is real and lies between ± 1, and n is a positive integer, provided we make the same assumption that underlies the ordinary process of finding DIRICHLET's forms for the Zonal Harmonic, viz. :—

$$\mathrm{I}_n(x) = \frac{2}{\pi}\int_{\cos^{-1}x}^{\pi} \sin(n-\tfrac{1}{2})\phi \sqrt{\{2(x-\cos\phi)\}}\, d\phi. \quad [\text{HEINE, §11}] \quad . \quad (15\text{C}).$$

We have (p. 453),

$$(1-x^2)\frac{d^2\mathrm{I}_n(x)}{dx^2} + n(n-1)\mathrm{I}_n(x) = 0.$$

From this equation we obtain the forms (9), (9A), (9B), (10), directly.

1st. Put

$$\mathrm{I}_n(x) = x^m z, \quad x^2 = \frac{1}{t};$$

then

$$t(1-t)\frac{d^2z}{dt^2} + [\tfrac{3}{2} - m - (\tfrac{3}{2} - m)t]\frac{dz}{dt} - \frac{m(m-1)}{4}z = 0 \quad . \quad . \quad . \quad (7\text{A}),$$

if

$$(m-n)(m+n-1) = 0;$$

of this

$$\mathrm{F}\left(-\frac{m}{2}, \frac{1}{2} - \frac{m}{2}, \frac{3}{2} - m, t\right)$$

is a solution.

2nd. Put

$$x^2 = t;$$

$$t(1-t)\frac{d^2\mathrm{I}_n}{dt^2} + \left(\frac{1}{2} - \frac{t}{2}\right)\frac{d\mathrm{I}_n}{dt} + \frac{n(n-1)}{4}\mathrm{I}_n = 0 \quad . \quad . \quad . \quad . \quad . \quad (7\text{B}),$$

of which

$$\mathrm{F}\left(-\frac{n}{2}, \frac{n}{2} - \frac{1}{2}, \frac{1}{2}, t\right)$$

is a solution.

3rd. Put

$$\mathrm{I}_n = xz, \quad x^2 = t,$$

$$t(1-t)\frac{d^2z}{dt^2} + \left(\frac{3}{2} - \frac{3t}{2}\right)\frac{dz}{dt} + \frac{n(n-1)}{4}z = 0 \quad . \quad . \quad . \quad . \quad . \quad (7\text{C}),$$

of which

$$F\left(-\frac{n}{2}+\frac{1}{2}, \frac{n}{2}, \frac{3}{2}, t\right)$$

is a solution.

4th. Put

$$\xi = x - \sqrt{(x^2-1)}, \quad I_n = \xi^m z, \quad t = \xi^2.$$

Then

$$t(1-t)\frac{d^2z}{dt^2} + [m+\tfrac{3}{2} - (m+\tfrac{1}{2})t]\frac{dz}{dt} + \frac{m}{2}z = 0 \quad . \quad . \quad . \quad . \quad (7\text{D})$$

if

$$(m+n)(m-n+1) = 0.$$

Of this

$$F(-\tfrac{1}{2}, \quad m, \quad m+\tfrac{3}{2}, \quad t)$$

is a solution.

From the differential equation (7), we may form a conception of $I_n(x)$, where n is not restricted to integral values; the discussion is found in Chapter III., where reasons are given for choosing (11), or its derived and equally general forms (14), (15), (15A), or (15B) as the definition of $I_n(x)$. These are identical with (9), (10), when n is a positive integer, and not otherwise.

$I_n(\pm 1) = 0$. This is obvious from (11). When n is a positive integer all the roots of $I_n(x) = 0$ are real, different, and lie between ± 1, and excepting zero, occur in pairs of the form $\pm\alpha$. This may be seen from (12). [HEINE, § 7.]

The sequence equation

$$(n+1)I_{n+1}(x) - (2n-1)xI_n(x) + (n-2)I_{n-1}(x) = 0 \quad . \quad . \quad (16)$$

connects three consecutive functions; this follows readily from (15B); n is not necessarily an integer.

We also find—a result required hereafter—

$$x^2 I_n(x) = \delta_n I_{n+2}(x) + \epsilon_n I_n(x) + \zeta_n I_{n-2}(x) \quad . \quad . \quad . \quad . \quad (17),$$

where

$$\delta_n = \frac{(n+1)(n+2)}{(2n-1)(2n+1)},$$

$$\epsilon_n = \frac{2n^2-2n-3}{(2n+1)(2n-3)},$$

$$\zeta_n = \frac{(n-2)(n-3)}{(2n-1)(2n-3)};$$

exceptions,

$$\zeta_1 = 0, \qquad \zeta_0 = 0.$$

We have for arbitrary n

$$P_n(x) = \frac{2}{\pi}\int_0^{\cos^{-1}x} d\phi \cdot \frac{\cos(n+\frac{1}{2})\phi}{\sqrt{\{2(\cos\phi - x)\}}}.$$

Whence, comparing with (15B),

$$\frac{dI_n(x)}{dx} = P_{n-1}(x) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18).$$

Other relations, of use in transformations, may be introduced here

$$(2n-1)\, I_n = P_n - P_{n-2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19),$$

$$I_n = -P_n + 2xP_{n-1} - P_{n-2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20),$$

$$\begin{aligned}(1-x^2)\frac{dI_n}{dx} &= (n-1)\, x\, I_n - (n+1)\, I_{n+1} \\ &= -nx\, I_n + (n-2)\, I_{n-1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21).\end{aligned}$$

These hold without restriction upon n.

We see from (18) that

$$I_n(x) = \frac{1.3\ldots 2n-3}{1.2\ldots n} \cdot \mathfrak{P}_1^{(n-1)}(x),$$

where $\mathfrak{P}_1^{(n-1)}(x)$ is defined and discussed by HEINE (Chapter III., §§ 30–33, and Chapter IV.), who obtains equivalents of the following results :—(9), (10), in § 51 ; (12) in § 31 ; (14), (14A) in § 50. Also the following, which are proved below :— (28), (32) in § 51 ; (33) and equivalents of (34), (35) in § 52 ; (22), (23) in § 62.

These might, therefore, have been assumed and the theory based upon them, but it seemed preferable to indicate briefly the direct development.

CHAPTER II.—DEVELOPMENTS IN TERMS OF I (x).

From equation (7) we find

$$(m-n)(m+n-1)\int_{-1}^{+1}\frac{I_m(x)\, I_n(x)}{1-x^2}\,dx = \left[I_m(x)\frac{dI_n(x)}{dx} - I_n(x)\frac{dI_m(x)}{dx}\right]_{-1}^{+1} \quad (22).$$

Whence

$$\int_{-1}^{+1}\frac{I_m(x)\, I_n(x)}{1-x^2}\,dx = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22\text{A}),$$

when $m \neq n$, and neither of them is 0 or 1.

The limiting value of this integral when $m = n$ is

$$\int_{-1}^{+1}\frac{I_n(x)\, I_n(x)}{1-x^2}\,dx = \frac{1}{2n-1}\left[\frac{dI_n}{dn}\cdot P_{n-1} - I_n\frac{dP_{n-1}}{dn}\right]_{-1}^{+1} = \frac{1}{2n-1}\cdot\left[\frac{dI_n}{dn}\cdot P_{n-1}\right]_{-1}^{+1}.$$

Using (15B),

$$\frac{dI_n}{dn} = \frac{2}{\pi}\int_0^{\cos^{-1}x}\phi\,.\,\sin\left(n-\tfrac{1}{2}\right)\phi\,\sqrt{\{2(\cos\phi - x)\}}\,d\phi.$$

This vanishes at the upper limit, and at the lower gives us

$$\frac{2}{\pi}\int_0^\pi \phi \,.\, \sin\left(n - \tfrac{1}{2}\right)\phi \,.\, 2\cos\frac{\phi}{2} \,.\, d\phi.$$

Integrating, we get the required expression

$$-\frac{\mathrm{P}_{n-1}(-1)}{2n-1}\,.\left\{\frac{2\cos n\pi}{n(n-1)} - \frac{2\sin n\pi\,.\,(2n-1)}{n^2(n-1)^2}\right\}$$

and when n is a positive integer

$$\int_{-1}^{-1}\frac{\mathrm{I}_n(x)\,\mathrm{I}_n(x)}{1-x^2}\,dx = \frac{2}{n(n-1)(2n-1)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23).$$

Every member of the set of functions, $\mathrm{I}_2(x)$, $\mathrm{I}_3(x)$, . . . $\mathrm{I}_n(x)$ . . . , vanishes with $x + 1$ and $x - 1$; besides, $\mathrm{I}_n(x)$ has $n - 2$ real factors lying between these two; and any two members of the set obey (22A). Now, if we have another function, $\phi(x)$, also vanishing with $x + 1$ and $x - 1$, it has been shown* that we can find a linear function of the I's, $\sum\limits_2^\infty \mathrm{A}_n \mathrm{I}_n(x)$, which shall be equal to $\phi(x)$ for all values of x between $+ 1$ and $- 1$, and

$$\mathrm{A}_n = \int_{-1}^{+1} dx\,.\,\phi(x)\,.\,\mathrm{I}_n(x)/(1-x^2)\Big/\int_{-1}^{+1} dx\,.\,\mathrm{I}_n(x)\,\mathrm{I}_n(x)/(1-x^2).$$

Or,

$$\phi(x) = \sum_{n=2}^{n=\infty}\frac{n(n-1)(2n-1)}{2}\,.\int_{-1}^{+1} dy\,\phi(y)\,\frac{\mathrm{I}_n(y)\,\mathrm{I}_n(x)}{1-y^2}, \quad . \quad . \quad . \quad (24)$$

where

$$\phi(\pm 1) = 0, \qquad \text{and} \qquad 1 > x > -1$$

It is easy to prove that this development is unique; I shall now show when it is convergent.

From the sequence equation, (16), we find

$$\begin{aligned}(n+1)\,n\,(n-1)\,[\mathrm{I}_{n+1}(x)\,\mathrm{I}_n(y) - \mathrm{I}_n(x)\,\mathrm{I}_{n+1}(y)]&\\ - n\,(n-1)\,(n-2)\,[\mathrm{I}_n(x)\,\mathrm{I}_{n-1}(y) - \mathrm{I}_{n-1}(x)\,\mathrm{I}_n(y)]&\\ = (x-y)\,.\,n\,(n-1)\,(2n-1)\,\mathrm{I}_n(x)\,\mathrm{I}_n(y)&\end{aligned}$$

from $n = 2$, onwards; and, therefore, the sum of the series (24) to n terms is

$$\tfrac{1}{2}\int_{-1}^{+1} dy\,.\,\phi(y)\,.\,(n+1)\,n\,(n-1)\,.\,\frac{\mathrm{I}_{n+1}(x)\,\mathrm{I}_n(y) - \mathrm{I}_n(x)\,\mathrm{I}_{n+1}(y)}{(x-y)(1-y^2)}.$$

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* 'Liouville,' vol. 1. The method is quoted, substantially, by Lord **Rayleigh**, 'Theory of Sound,' vol. 1, ch. 6.

Similarly, it may be shown that the sum of n terms of the development of $f(x)$ in Zonal Harmonics is

$$\tfrac{1}{2}\int_{-1}^{+1} dy\, f(y)\,(n+1)\,\frac{\mathrm{P}_{n+1}(x)\,\mathrm{P}_n(y) - \mathrm{P}_n(x)\,\mathrm{P}_{n+1}(y)}{x-y}.$$

And it is shown by HEINE (vol. 1, § 119), that the limit of this, when n is infinite is the mean $\tfrac{1}{2}\{f(x+0)+f(x-0)\}$, provided $f(x)$ is not a function with an infinite number of maxima and minima.

Moreover, (HEINE, vol. 1, § 40), when n is very great

$$\mathrm{P}_n(x) = \frac{\xi^{-n}(1-\xi^2)^{-\frac{1}{2}}}{\pi^{\frac{1}{2}}n^{\frac{1}{2}}},$$

and it will be shown, (p. 476), when n is very great

$$\mathrm{I}_n(x) = \frac{\xi^{-n}(1-\xi^2)^{\frac{1}{2}}}{2\pi^{\frac{1}{2}}n^{\frac{3}{2}}},$$

where $\xi = x - \sqrt{(x^2-1)}$, as on p. 456.

Therefore, the two integrals above tend respectively to the same limits, as

$$\frac{1}{2\pi}\int_{-1}^{+1}\phi(y)\,\frac{(\xi\eta)^{-n}}{\xi\eta-1}\,\frac{(1-\xi^2)^{\frac{1}{2}}}{(1-\eta^2)^{\frac{1}{2}}}\,d\eta,$$

$$\frac{1}{2\pi}\int_{-1}^{+1}f(y)\,.\,\frac{(\xi\eta)^{-n-1}}{\xi\eta-1}\,\frac{\xi}{\eta}\,\frac{(1-\eta^2)^{\frac{1}{2}}}{(1-\xi^2)^{\frac{1}{2}}}\,d\eta,$$

where $\eta = y - \sqrt{(y^2-1)}$.

The limit of the second we know, and if we write

$$f(y) = \phi(y)\,\frac{\eta}{\xi}\,\frac{1-\xi^2}{1-\eta^2} = \phi(y)\,\frac{\sqrt{(1-x^2)}}{\sqrt{(1-y^2)}},$$

the second reduces to the first. Therefore the limit of the first is the mean value of $\phi(x)$ at the point x, provided $\phi(x)/\sqrt{(1-x^2)}$ is a finite function of x, which does not possess an infinite number of maxima and minima.

We find the function

$$\tfrac{1}{2} + \frac{x}{2} + \sum_{r=1}^{r=\infty}\mathrm{I}_{2r}(x)\,\frac{4r-1}{2}\,.\,\frac{n(n-2)\ldots(n-2r+2)}{(n+2r-1)(n+2r-3)\ldots(n+1)}$$
$$+ \sum_{r=1}^{r=\infty}\mathrm{I}_{2r+1}(x)\,\frac{4r+1}{2}\,.\,\frac{(n-1)(n-3)\ldots(n-2r+1)}{(n+2r)(n+2r-2)\ldots(n+2)} \quad . \quad . \qquad (25)$$

is one which is equal to x^n when x is positive and to zero when x is negative, for all values of n.

When n is a positive integer

$$x^n = \frac{1.2\ldots n}{1.3\ldots 2n-1}\cdot\left[(2n-1)\,\mathrm{I}_n(x) + (2n-5)\,\frac{2n-1}{2}\cdot\mathrm{I}_{n-2}(x)\right.$$
$$\left. + (2n-9)\,\frac{(2n-1)(2n-3)}{2.4}\cdot\mathrm{I}_{n-4} + \ldots\right] \quad (25\text{A}),$$

whether x is positive or negative; this equation is in fact an identity.

And if

$$\mathrm{F}(x) = c_0 + c_1x + c_2x^2 + \ldots$$

we may write

$$\mathrm{F}(x) = b_0\mathrm{I}_0(x) + b_1\mathrm{I}_1(x) + c_2\mathrm{I}_2(x) + \ldots,$$

where

$$b_n = \frac{1.2\ldots n}{1.3\ldots 2n-3}\left[c_n + \frac{(n+1)(n+2)}{2(2n+1)}\,c_{n+2}\right.$$
$$\left. + \frac{(n+1)(n+2)(n+3)(n+4)}{2.4.(2n+1)(2n+3)}\,c_{n+4}\ldots\right] \ldots \quad (25\text{B})$$

from $n = 2$, onwards, while

$$-b_0 + b_1 = \mathrm{F}(1)\,; \quad -b_0 - b_1 = \mathrm{F}(-1). \quad [\text{Heine, § 16.}]$$

We have

$$\cos m\theta = \cos\frac{m\pi}{2}\cos m\left(\frac{\pi}{2}-\theta\right) + \sin\frac{m\pi}{2}\sin m\left(\frac{\pi}{2}-\theta\right),$$

$$\sin m\theta = \sin\frac{m\pi}{2}\cos m\left(\frac{\pi}{2}-\theta\right) - \cos\frac{m\pi}{2}\sin m\left(\frac{\pi}{2}-\theta\right);$$

and if $\cos\theta = x$, we find

$$\cos m\left(\frac{\pi}{2}-\theta\right)$$
$$= \cos\frac{m\pi}{2}\left[-\mathrm{I}_0(x) - 3\cdot\frac{m^2}{m^2-1^2}\cdot\mathrm{I}_2(x) + 7\cdot\frac{m^2\,.\,m^2-2^2}{m^2-1^2\,.\,m^2-3^2}\cdot\mathrm{I}_4(x) - \ldots\right]$$

$$\sin m\left(\frac{\pi}{2}-\theta\right)$$
$$= \sin\frac{m\pi}{2}\left[\mathrm{I}_1(x) - 5\cdot\frac{m^2-1^2}{m^2-2^2}\cdot\mathrm{I}_3(x) + 9\cdot\frac{m^2-1^2\,.\,m^2-3^2}{m^2-2^2\,.\,m^2-4^2}\cdot\mathrm{I}_5(x) - \ldots\right] \quad . \ (26).$$

When m is a positive integer

$$\cos m\theta = \frac{2 \,.\, 4 \ldots 2m - 2}{1 \,.\, 3 \ldots 2m - 1} \Big[(2m - 1)\, \mathrm{I}_m (\cos\theta)$$
$$+ \frac{1 \,.\, 2m - 1}{2 \,.\, 2m - 2} (2m - 5) \,.\, \mathrm{I}_{m-2} (\cos\theta)$$
$$+ \frac{1 \,.\, 3 \,.\, 2m - 1 \,.\, 2m - 3}{2 \,.\, 4 \,.\, 2m - 2 \,.\, 2m - 4} (2m - 9)\, \mathrm{I}_{m-4} (\cos\theta) + \ldots \Big] \quad . \quad (26\text{A}).$$

Where m is even, m^2 occurs in the numerator; where m is odd, m is absent from the denominator. [HEINE, § 19.]

We have, by (19)

$$\mathrm{P}_n(x) = (2n - 1)\, \mathrm{I}_n(x) + \mathrm{P}_{n-2}(x) = (2n - 1)\, \mathrm{I}_n(x) + (2n - 5)\, \mathrm{I}_{n-2}(x) + \ldots$$

Hence, if we can effect the development

$$\mathrm{F}(x) = c_0 \mathrm{P}_0(x) + c_1 \mathrm{P}_1(x) + c_2 \mathrm{P}_2(x) + \ldots,$$

we can write

$$\mathrm{F}(x) = b_0 \mathrm{I}_0(x) + b_1 \mathrm{I}_1(x) + b_2 \mathrm{I}_2(x) + \ldots,$$

where

$$b_n = (2n - 1)(c_n + c_{n+2} + \ldots) \quad . \; . \; . \; . \; . \; . \; . \; . \quad (27).$$

If we develop $(1 - yx)/(y - x)$ in ascending powers of x, and then transform by (25B), we get

$$\frac{1 - yx}{y - x} = - x + \sum_{n=2}^{n=\infty} n(n - 1)(2n - 1)\, \mathrm{H}_n(y)\, \mathrm{I}_n(x),$$

where

$$\mathrm{H}_n(y) = - \frac{\Pi(n - 2)\, \Pi(-\tfrac{1}{2})}{2^n \Pi(n - \tfrac{1}{2})}\, y^{-n+1}\, \mathrm{F}\left(\frac{n - 1}{2}, \frac{n}{2}, n + \tfrac{1}{2}, y^{-2}\right) \quad . \quad (28).$$

The convergency of this development is discussed on p. 476.

In future references to the expression $\mathrm{H}_n(y)$ we shall not in general suppose n to be a positive integer. On comparing it with the expression (9) for $\mathrm{I}_n(x)$, it is seen that the former may be derived from the latter by changing x into y, n into $- n + 1$, and multiplying by a constant factor.

Since a similar change transforms

$$(1 - x^2) \frac{d^2 z}{dx^2} + n(n - 1) z = 0 \quad \text{into} \quad (1 - y^2) \frac{d^2 z}{dy^2} + n(n - 1) z = 0,$$

we see that $\mathrm{H}_n(x)$ is another solution of (7), a result at which we have already arrived on p. 459.

There are two independent solutions to the equation (7), whatever the value of the independent variable may be, but (28) ceases to be a permissible expression for one of them when mod. $y^2 < 1$.

For mod. $y^2 = 1$, the series is convergent, since $(n-1)/2 + n/2 - (n + \frac{1}{2})$ is negative, and we have, by the well-known expression for the sum of a hypergeometric series of which the fourth element is unity,

$$\mathrm{H}_n(1) = -\frac{1}{n(n-1)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29).$$

The function $\mathrm{H}_n(x)$ is of odd degree in x when n is even, and of even degree when n is odd; it vanishes when x is infinite. For $n = 0$ or 1 the given expression fails, but if H is still a solution of (7), different from the corresponding I, we may put

$$\mathrm{H}_0(x) = x; \qquad \mathrm{H}_1(x) = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30).$$

Certain functions which are not finite for all values of x between ± 1, may be developed in a series involving $\mathrm{H}(x)$ in place of $\mathrm{I}(x)$.

Thus

$$\frac{1}{x-y} = \sum_2^\infty n(n-1)(2n-1)\,\mathrm{H}_n(x)\,\frac{\mathrm{I}_n(y)}{1-y^2} = \sum_2^\infty (2n-1)\,\mathrm{H}_n(x)\,\frac{d^2\,\mathrm{I}_n(y)}{dy^2};$$

differentiate $n - 2$ times with respect to y, and then put $y = 0$, and we find

$$x^{-n+1} = -\frac{1.3\ldots 2n-3}{1.2\ldots n-2}\Big[(2n-1)\,\mathrm{H}_n(x) - (2n+3)\,\frac{2n-1}{2}\,\mathrm{H}_{n+2}(x)$$
$$+ (2n+7)\,\frac{2n-1.2n+1}{2.4}\,\mathrm{H}_{n+4}(x) - \ldots\Big] \quad (31),$$

and if

$$\mathrm{F}(x) = c_1x^{-1} + c_2x^{-2} + c_3x^{-3} + \ldots,$$

we may re-write it

$$= b_2\mathrm{H}_2(x) + b_3\mathrm{H}_3(x) + \ldots,$$

where

$$b_n = -\frac{1.3\ldots 2n-6}{1.2\ldots n-2}\cdot\Big[c_{n-1} - \frac{n-2.n-3}{2.2n-3}\,c_{n-3}$$
$$+ \frac{n-2.n-3.n-4.n-5}{2.4.2n-3.2n-5}\,c_{n-5} - \ldots\Big] \quad (31\text{A}),$$

provided the expansion should prove convergent. [HEINE, § 21.]

A thorough discussion of the function H is given in the next chapter.

CHAPTER III.—ON THE FUNCTION $H_n(x)$.

The function $H_n(x)$ which, when mod. x is not less than 1, may be written

$$H_n(x) = -2^{-n}\frac{\Pi(n-2)\,\Pi(-\frac{1}{2})}{\Pi(n-\frac{1}{2})}\,x^{-n+1}\,F\left(\frac{n-1}{2},\frac{n}{2},n+\tfrac{1}{2},x^{-2}\right) \quad . \quad . \quad (28),$$

may be identified in any expression, in which mod. x may be as great as we please, as that solution of (7), which vanishes when mod. x is infinite, and is equal to $-\frac{1}{n(n-1)}$ when $x = 1$.

Thus, reverting to the expressions of p. 460, we see that, introducing a constant multiplier,

$$H_n(x) = -\frac{\Pi(n-2)\,\Pi(\frac{1}{2})}{\Pi(n-\frac{1}{2})}\,\xi^{n-1}\,F(-\tfrac{1}{2},n-1,n+\tfrac{1}{2},\xi^2) \quad . \quad . \quad . \quad (32).$$

Again, by a well-known theorem

$$\frac{\Pi(n-2)\,\Pi(\frac{1}{2})}{\Pi(n-\frac{1}{2})}\,F(-\tfrac{1}{2},n-1,n+\tfrac{1}{2},\xi^2) = \int_0^1 u^{n-2}(1-u)^{\frac{1}{2}}(1-u\xi^2)^{\frac{1}{2}}\,du.$$

Hence

$$H_n(x) = -\xi^{n-1}\int_0^1 u^{n-2}(1-u)^{\frac{1}{2}}(1-u\xi^2)^{\frac{1}{2}}\,du \quad . \quad . \quad . \quad , \quad (33),$$

a form which may be compared with (11).

In (32) and (33) no restriction is placed upon the value of x, and n is supposed to be a real positive quantity, but not necessarily an integer.

From (33) we get a form analogous to (14).

For write $u\xi = x - \sqrt{(x^2-1)}\cosh\phi$; this gives, on substitution,

$$H_n(x) = -\int_0^{\phi_0} d\phi\,(x^2-1)\sinh^2\phi\,\{x-\sqrt{(x^2-1)}\cosh\phi\}^{n-2} \quad . \quad . \quad (34),$$

where

$$\phi_0 = \tfrac{1}{2}\log\frac{x+1}{x-1}.$$

The sign which we attach to the logarithm is immaterial; I shall take that sign which makes the real part, if any, positive; or, as we may write it,

$$\phi_0 = \tfrac{1}{2}\log\frac{x+1}{x-1} = \log\rho - i\sigma,$$

$$\rho > 1, \quad \text{and} \quad -\tfrac{1}{2}\pi < \sigma < \tfrac{1}{2}\pi.$$

In (34) write

$$(x - \sqrt{(x^2 - 1)} \cosh \phi)(x + \sqrt{(x^2 - 1)} \cosh \theta) = 1\,.$$

Substitute, and we get

$$\mathrm{H}_n(x) = -\int_0^\infty \frac{(x^2 - 1) \sinh^2 \theta \, d\theta}{(x + \sqrt{x^2 - 1}\,.\cosh \theta)^{n+1}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

In this expression, x may not be negative and greater than unity.

The expression (34) may be made to furnish us with another, on the model of MEHLER's form for $\mathrm{P}_n(x)$.

For, write

$$x - \sqrt{(x^2 - 1)} \cosh \phi = e^\chi.$$

Then

$$\sqrt{(x^2 - 1)} \sinh \phi = \sqrt{(e^{2\chi} - 2xe^\chi + 1)},$$

where the square root on the right has such a sign that its real part is of like sign with the real part of $\sqrt{(x^2 - 1)} \sinh \phi$.

Thus

$$\mathrm{H}_n(x) = -\int_{-\infty}^{\cosh^{-1} x} d\chi \sqrt{\{2(\cosh \chi - x)\}}\, e^{(n - \frac{1}{2})\chi} \quad . \quad . \quad . \quad . \quad (36),$$

where

$$\sinh(\cosh^{-1} x) = -\sqrt{(x^2 - 1)}.$$

This expression corresponds to (15), p. 458; (15) and (36) may be differentiated once with respect to x, but not twice.

We have, by p. 465, where n is a positive integer,

$$\sum_{n=2}^{n=\infty} n(n-1)(2n-1)\,\mathrm{H}_n(y)\,\mathrm{I}_n(x) = \frac{1 - yx}{y - x} + x = \frac{1 - x^2}{y - x} \quad . \quad . \quad (37).$$

Hence, multiplying by $y - x$,

$$-2\mathrm{I}_2(x)$$
$$= \sum_{n=2}^{n=\infty} n(n-1)\Big[(2n-1)y\,\mathrm{H}_n(y)\mathrm{I}_n(x) - \mathrm{H}_n(y)\{(n+1)\,\mathrm{I}_{n+1}(x) - (n-2)\,\mathrm{I}_{n-1}(x)\}\Big],$$

since

$$(2n-1)\,x\mathrm{I}_n(x) = (n+1)\,\mathrm{I}_{n+1}(x) + (n-2)\,\mathrm{I}_{n-1}(x) \quad . \quad . \quad . \quad (16).$$

Hence we get

$$(n+1)\,\mathrm{H}_{n+1}(y) - (2n-1)\,y\mathrm{H}_n(y) + (n-2)\,\mathrm{H}_{n-1}(y) = 0 \quad . \quad . \quad (38)$$

$$3\mathrm{H}_3(y) - 3y\mathrm{H}_2(y) + 1 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38\text{A}).$$

A proof of (38) which is valid when n is not a positive integer, may be obtained from (36).

From (38) it follows that

$$x^2 H_n(x) = \delta_n H_{n+2}(x) + \epsilon_n H_n(x) + \zeta_n H_{n-2}(x) \quad . \quad . \quad . \quad . \quad (39),$$

where δ_n, ϵ_n, ζ_n are the quantities so called on p. 460; with the following exceptions

$$\left.\begin{aligned} x^2 H_3(x) &= \delta_3 H_5(x) + \epsilon_3 H_3(x) - \tfrac{1}{15} \\ x^2 H_2(x) &= \delta_2 H_4(x) + \epsilon_2 H_2(x) - \frac{x}{3} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (39\text{A}).$$

If $Q_n(x)$ be the Zonal Harmonic of the second kind

$$Q_n(x) = \int_0^{\phi_0} d\phi \, \{x - \sqrt{(x^2 - 1)} \cosh \phi\}^n \, d\phi,$$

where

$$\phi_0 = \tfrac{1}{2} \log \frac{x+1}{x-1} \cdot \quad [\text{Heine}, \S\ 36.]$$

From this may be derived, as on p. 458 for $I_n(x)$, the expression

$$Q_n(x) = \int_{-\infty}^{\cosh^{-1} x} \frac{e^{(n+\frac{1}{2})x} d\chi}{\sqrt{\{2(\cosh \chi - x)\}}}$$

where $\sinh(\cosh^{-1} x) = -\sqrt{(x^2 - 1)}$.

Hence

$$\frac{d\,H_n(x)}{dx} = Q_{n-1}(x) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (40),$$

and also

$$(2n - 1) H_n(x) = Q_n(x) - Q_{n-2}(x) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (41).$$

Again, as on p. 461, we find

$$H_n(x) = -Q_n(x) + 2x Q_{n-1}(x) - Q_{n-2}(x) \quad . \quad . \quad . \quad . \quad . \quad (42),$$

and

$$H_n(x) = -\frac{\Pi(n-2)\,\Pi(-\frac{1}{2})}{2^n \Pi(n - \frac{1}{2})} \mathfrak{O}_1^{(n)}(x),$$

where $\mathfrak{O}_1^{(n)}(x)$ is defined by Heine, § 31.

By means of the sequence equations

$$(n+1) I_{n+1}(x) - (2n - 1) x I_n(x) + (n - 2) I_{n-1}(x) = 0 \quad . \quad . \quad . \quad (16)$$

$$(n+1) H_{n+1}(x) - (2n - 1) x H_n(x) + (n - 2) H_{n-1}(x) = 0 \quad . \quad . \quad (38)$$

we see that

$$\left.\begin{aligned} \mathrm{I}_n(x) &= \alpha_r \mathrm{I}_{n-r}(x) - \beta_r \mathrm{I}_{n-r-1}(x) \\ \mathrm{H}_n(x) &= \alpha_r \mathrm{H}_{n-r}(x) - \beta_r \mathrm{H}_{n-r-1}(x) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (43),$$

where α_r, β_r are rational functions of degrees r, $r-1$ respectively in x. This holds whether n is integral or not. If n is a positive integer, we have, taking regard to the equations

$$\begin{aligned} 3\mathrm{I}_3(x) - 3x\mathrm{I}_2(x) \qquad &= 0 \\ 3\mathrm{H}_3(x) - 3x\mathrm{H}_2(x) + 1 &= 0 \end{aligned}$$

$$\left.\begin{aligned} \mathrm{I}_n(x) &= \alpha_{n-2}\mathrm{I}_2(x) \\ \mathrm{H}_n(x) &= \alpha_{n-2}\mathrm{H}_2(x) + \beta_{n-2} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (43\text{A}).$$

But by (34)

$$\mathrm{H}_2(x) = -\int_0^{\frac{1}{2}\log(x+1)/(x-1)} (x^2-1)\sinh^2\phi \,.\, d\phi = \tfrac{1}{2}\mathrm{I}_2(x)\log\frac{x+1}{x-1} - \frac{x}{2} \quad . \quad (44\text{A}).$$

Hence we get the expression, n being a positive integer,

$$\mathrm{H}_n(x) = \tfrac{1}{2}\,\mathrm{I}_n(x)\log\frac{x+1}{x-1} - \mathrm{K}_n(x) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (44),$$

where $\mathrm{K}_n(x)$ is a rational integral function of x, of degree $n-1$.

By substituting (44) in the equation (7)

$$(1-x^2)\frac{d^2\mathrm{H}_n(x)}{dx^2} + n(n-1)\,\mathrm{H}_n(x) = 0$$

we can obtain an expression for $\mathrm{K}_n(x)$ in terms of the I's, viz.,

$$\mathrm{K}_n(x) = \sum^{r=k} \frac{2(2n-4r+1)}{(2r-1)(n-r)}\left[1 - \frac{(2r-1)(n-r)}{n(n-1)}\right]\mathrm{I}_{n-2r+1} \quad . \quad . \quad (44\text{B}),$$

the series beginning with I_0 or I_1. [HEINE, § 26.]

The expression (44), where the logarithm is defined on p. 467, might be taken as the definition of $\mathrm{H}_n(x)$ when n is a positive integer, without restriction upon x; but when x is real and less than unity, it is advisable to modify it; viz., from (44) we get

$$\tfrac{1}{2}\,\mathrm{I}_n(x)\log\frac{1+x}{1-x} - \mathrm{K}_n(x) \pm \frac{i\pi}{2}\,.\,\mathrm{I}_n(x)\,.$$

In this case, I take

$$\mathrm{H}_n(x) = \tfrac{1}{2}\,\mathrm{I}_n(x)\log\frac{1+x}{1-x} - \mathrm{K}_n(x) \quad . \quad . \quad . \quad . \quad . \quad . \quad (44\text{C}),$$

a form which, along with $\mathrm{I}_n(x)$, is capable of expressing any solution of (7).

The form (44) is easily derived from

$$\mathrm{H}_n(x) = \tfrac{1}{2}\int_{-1}^{+1}\frac{\mathrm{I}_n(y)}{x-y}\,dy \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (44\text{D}),$$

which itself follows directly from the development (37). [HEINE, § 28.]

The two forms (33), (11), are simply transformations of the two solutions

$$y = \int_0^1 u^{\beta-1}(1-u)^{\gamma-\beta-1}(1-tu)^{-\alpha}\,du, \qquad \beta, \gamma-\beta, \text{ positive}$$

$$y = \int_1^{\frac{1}{t}} u^{\beta-1}(1-u)^{\gamma-\beta-1}(1-tu)^{-\alpha}\,du, \qquad \gamma-\beta, 1-\alpha, \text{ positive}$$

which, among others, JACOBI gives ('Crelle,' vol. 56) as solutions of

$$t(1-t)\frac{d^2y}{dt^2} + [\gamma - (\alpha+\beta+1)\,t]\frac{dy}{dt} - \alpha\beta y = 0.$$

See (7D) p. 460 ; $m = n - 1$ in the first, and $= -n$ in the second. $t = \xi^2$, and JACOBI supposes it positive, but here that is not necessary.

Let us now coordinate the various solutions that have been obtained.

The different series used are only permissible expressions for the solutions of (7) so long as they are convergent.

Now $\mathrm{F}(\alpha, \beta, \gamma, x)$ is convergent,

when $\alpha + \beta - \gamma$ is not positive, if mod. $x < 1$
when $\alpha + \beta - \gamma$ is negative, if mod. $x \not> 1$.

Hence

(9) (28) are permissible for mod. $x \not< 1$,
(9A) (9B) are permissible for mod. $x \not> 1$,
(10) (32) whatever x may be.

Now, by direct development, we know, ($\beta, \gamma - \beta$, positive),

$$\int_0^1 u^{\beta-1}(1-u)^{\gamma-\beta-1}(1-xu)^{-\alpha}\,du = \frac{\Pi(\beta-1)\,\Pi(\gamma-\beta-1)}{\Pi(\gamma-1)}\,\mathrm{F}(\alpha, \beta, \gamma, x).$$

Hence $(33) \equiv (32)$, and, therefore, $\equiv (31)$, when this last is a permissible expression.

But (11) $\not\equiv$ (9), or (10), unless n is a positive integer. For (11) $\equiv$ (14) and from

$$(14) \qquad \operatorname*{Lt}_{x=1} \frac{I_n(x)}{x^2-1} = \tfrac{1}{2}.$$

$$(9) \qquad \frac{I_n(x)}{x^2-1} = -2^{n-1}\frac{\Pi(n-\frac{3}{2})}{\Pi(n)\,\Pi(-\frac{1}{2})}\,x^n\,F\left(\tfrac{3}{2}-\frac{n}{2},\ 1-\frac{n}{2},\ \tfrac{3}{2}-n,\ x^{-2}\right).$$

$$(10) \qquad \frac{I_n(x)}{x^2-1} = 2\,\frac{\Pi(n-\frac{3}{2})}{\Pi(n)\,\Pi(-\frac{1}{2})}\,\xi^{-n+2}\,F\left(2-n,\ \tfrac{3}{2},\ \tfrac{3}{2}-n,\ \xi^2\right).$$

But the series on the right are divergent for $x = 1$. Similarly (9A) and (9B) do not agree with (11).

Taking the definitions (11) and (33) for $I_n(x)$ and $H_n(x)$, we find

$$I_n(x) = e^{in\pi}\,I_n(-x) \qquad H_n(x) = e^{i(n-1)\pi}\,H_n(-x) \quad \ldots\ldots \quad (A)$$

$$I_n(\pm 1) = 0 \qquad H_n(\pm 1) = -\frac{1}{n(n-1)} \quad \ldots\ldots \quad (B)$$

$$I_n(0) = \frac{2}{\pi}\,\frac{\Pi(\frac{1}{2}n-\frac{3}{2})\,\Pi(\frac{1}{2})}{\Pi(\frac{1}{2}n)}\,.\cos\frac{n\pi}{2} \qquad H_n(0) = \frac{\Pi(\frac{1}{2}n-\frac{3}{2})\,\Pi(\frac{1}{2})}{\Pi(\frac{1}{2}n)}\left(i\cos\frac{n\pi}{2} - \sin\frac{n\pi}{2}\right) \quad (C)$$

$$\operatorname*{Lt}_{x=\infty} x^{-n}\,I_n(x) = 2^{n-1}\,\frac{\Pi(n-\frac{3}{2})}{\Pi(n)\,\Pi(-\frac{1}{2})} \qquad \operatorname*{Lt}_{x=\infty} x^{n-1}\,H_n(x) = -2^{-n}\,\frac{\Pi(n-2)\,\Pi(-\frac{1}{2})}{\Pi(n-\frac{1}{2})} \quad (D)$$

Add to these, when n is a positive integer,

$$\left.\begin{array}{ll} n \text{ odd} & \displaystyle\operatorname*{Lt}_{x=0}\frac{I_n(x)}{x} = P_{n-1}(0) = (-1)^{\frac{1}{2}(n-1)}\,\frac{\Pi(\frac{1}{2}n-1)}{\Pi(\frac{1}{2}n-\frac{1}{2})\,\Pi(-\frac{1}{2})} \\ n \text{ even} & \displaystyle\operatorname*{Lt}_{x=0}\frac{H_n(x)}{x} = Q_{n-1}(0) = (-1)^{\frac{1}{2}n}\,\frac{\Pi(\frac{1}{2}n-1)\,\Pi(\frac{1}{2})}{\Pi(\frac{1}{2}n-\frac{1}{2})} \end{array}\right\} \quad \ldots\ldots \quad (E)$$

[HEINE, §§ 4, 25.]

These results will be quoted as . . . . (45)

We can now give the expression for $H_n(x)$ in ascending powers of x. For the expressions (9A) and (9B) satisfy the equation (7), whether n is odd or even; and where $I_n(x)$ can be developed in even powers of x, $H_n(x)$ can be developed in odd powers, and conversely. The constant factor is determined by (45); hence

n even

$$H_n(x) = (-1)^{\frac{1}{2}n} \frac{\Pi(\frac{1}{2}n - 1)\,\Pi(\frac{1}{2})}{\Pi(\frac{1}{2}n - \frac{1}{2})}\, x F\left(-\frac{n}{2} + \tfrac{1}{2}, \frac{n}{2}, \tfrac{3}{2}, x^2\right) \quad . \quad . \quad . \quad (28\text{A}),$$

n odd

$$H_n(x) = (-1)^{\frac{1}{2}(n+1)} \frac{\Pi(\frac{1}{2}n - \frac{3}{2})\,\Pi(\frac{1}{2})}{\Pi(\frac{1}{2}n)}\, F\left(-\frac{n}{2}, \frac{n}{2} + \tfrac{1}{2}, \tfrac{1}{2}, x^2\right) \quad . \quad . \quad (28\text{B}).$$

These apply when mod. $x \not> 1$; and only when n is integral; for we see from (45A) that neither $I_n(x)$ nor $H_n(x)$ can be expanded in a series of integral powers of x when n is not an integer.

Two functions, R_n, T_n, have been used by Mr. HICKS[*] in connection with his researches on the motion of circular vortices, which are such that

$$R_n = \frac{4n^2 - 1}{4} I_{n+\frac{1}{2}}(\cosh u),$$

$$T_n = -\frac{4n^2 - 1}{4} H_{n+\frac{1}{2}}(\cosh u).$$

Mr. BASSET[†] employs a function L_n, where

$$L_n(\xi) = (-1)^n \frac{2}{\pi} \cdot \xi^{\frac{1}{2}} H_{n+\frac{1}{2}}(x),$$

and finds the sequence equation corresponding to (38), and the values of L_0, L_1 in elliptic integrals.

Mr. HICKS gives equations which are equivalent to (18) and (40); approximate expressions for the first five R's and first four T's in terms of $k = e^{-u}$ (or, if we write $x = \cosh u$, k is the ξ of p. 456); and an expression for T_n, which leads to

$$H_{n+\frac{1}{2}}(x) = \int_0^\pi \cos n\theta \sqrt{\{2(x - \cos\theta)\}}\, d\theta \quad . \quad . \quad . \quad . \quad . \quad (46).$$

A proof of (46) will be given in the next chapter (p. 483); but I will here show that it applies only when n is a positive integer.

For

$$H_{n+\frac{3}{2}}(x) + H_{n-\frac{1}{2}}(x) = 2xH_{n+\frac{1}{2}}(x) - 2\sqrt{2}\int_0^\pi \cos n\theta\,(x - \cos\theta)^{\frac{3}{2}}\, d\theta.$$

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[*] 'Phil. Trans.,' 1884, 1885.

[†] 'Hydrodynamics,' vol. 1, § 115.

Now

$$2\sqrt{2}\int_0^\pi \cos n\theta\,(x-\cos\theta)^{\frac{3}{2}}\,d\theta = \left[\frac{2\sqrt{2}}{n}\sin n\theta\,(x-\cos\theta)^{\frac{3}{2}}\right]_0^\pi + \frac{3}{2n}\{\mathrm{H}_{n+\frac{3}{2}}(x)-\mathrm{H}_{n-\frac{1}{2}}(x)\}.$$

Now, *if n is a positive integer*, the expression in [ ] vanishes at both limits, and we find the sequence equation

$$(n+\tfrac{3}{2})\,\mathrm{H}_{n+\frac{3}{2}}(x) - 2nx\mathrm{H}_{n+\frac{1}{2}}(x) + (n-\tfrac{3}{2}).\mathrm{H}_{n-\frac{1}{2}}(x) = 0 \quad . \quad . \quad . \quad (38).$$

We see from p. 470 that we may write

$$\mathrm{I}_{n+\frac{1}{2}}(x) = \alpha_{n-1}\mathrm{I}_{\frac{3}{2}}(x) - \beta_{n-1}\mathrm{I}_{\frac{1}{2}}(x)$$
$$\mathrm{H}_{n+\frac{1}{2}}(x) = \alpha_{n-1}\mathrm{H}_{\frac{3}{2}}(x) - \beta_{n-1}\mathrm{H}_{\frac{1}{2}}(x),$$

where, n being a positive integer, α_{n-1}, β_{n-1} are rational integral functions of x of degrees $n-1$, $n-2$, respectively; and $\mathrm{I}_{\frac{3}{2}}$, $\mathrm{I}_{\frac{1}{2}}$, $\mathrm{H}_{\frac{3}{2}}$, $\mathrm{H}_{\frac{1}{2}}$ may be expressed in terms of complete elliptic integrals of the first and second kinds.

For we have, by (14A)

$$\begin{aligned}\mathrm{I}_{\frac{1}{2}}(x) &= \frac{1}{\pi}\int_0^\pi \frac{(x^2-1)\sin^2\theta\,d\theta}{(x+\sqrt{x^2-1}\,.\cos\theta)^{\frac{3}{2}}}\\ &= -\frac{2}{\pi}\int_0^\pi \frac{\sqrt{x^2-1}\,.\cos\theta\,d\theta}{(x+\sqrt{x^2-1}\cos\theta)^{\frac{1}{2}}}\\ &= -\frac{2}{\pi}\int_0^\pi\left\{(x+\cos\theta\sqrt{x^2-1})^{\frac{1}{2}} - \frac{x}{(x+\cos\theta\sqrt{x^2-1})^{\frac{1}{2}}}\right\}d\theta,\end{aligned}$$

and

$$x+\cos\theta\sqrt{(x^2-1)} = \frac{1}{\xi}\left\{1-k'^2\sin^2\frac{\theta}{2}\right\}$$

where

$$k'^2 = 1-\xi^2,$$

$$\therefore\ \mathrm{I}_{\frac{1}{2}}(x) = -\frac{2}{\pi}\left[\xi^{-\frac{1}{2}}\,\mathrm{E}\left(k',\frac{\pi}{2}\right) - x\xi^{\frac{1}{2}}\mathrm{F}\left(k',\frac{\pi}{2}\right)\right] \quad . \quad . \quad . \quad (47).$$

Also, by (14)

$$\begin{aligned}\mathrm{I}_{\frac{3}{2}}(x) &= \frac{1}{\pi}\int_0^\pi \frac{(x^2-1)\sin^2\theta\,d\theta}{(x+\cos\theta\sqrt{x^2-1})^{\frac{1}{2}}}\\ &= \frac{2}{\pi}\int_0^\pi \sqrt{x^2-1}\cos\theta\,(x+\cos\theta\sqrt{x^2-1})^{\frac{1}{2}}\,d\theta,\end{aligned}$$

and

$$\begin{aligned}(x^2-1)\sin^2\theta = &-\sqrt{x^2-1}\cos\theta\,(x+\sqrt{x^2-1}\,.\cos\theta)\\ &+ x(x+\sqrt{x^2-1}\,.\cos\theta) - 1\end{aligned}$$

$$\therefore \quad \mathrm{I}_{\frac{3}{2}}(x) = -\tfrac{1}{2}\mathrm{I}_{\frac{3}{2}}(x) + \frac{x}{\pi}\int_0^\pi (x + \cos\theta\sqrt{x_2 - 1})^{\frac{1}{2}}\,d\theta$$
$$- \frac{1}{\pi}\int_0^\pi \frac{d\theta}{(x + \cos\theta\sqrt{x^2 - 1})^{\frac{1}{2}}},$$

or

$$\mathrm{I}_{\frac{3}{2}}(x) = \frac{2}{3\pi}\left[x\xi^{-\frac{1}{2}}\,\mathrm{E}\left(k', \frac{\pi}{2}\right) - \xi^{\frac{1}{2}}\mathrm{F}\left(k', \frac{\pi}{2}\right)\right] \quad . \quad . \quad . \quad . \quad . \qquad (47\text{A}).$$

Also, using the results given by Mr. BASSET ('Hydrodynamics,' vol. 1, p. 108), we have

$$\mathrm{H}_{\frac{1}{2}}(x) = \quad 2\left[\xi^{-\frac{1}{2}}\,\mathrm{E}\left(\xi, \frac{\pi}{2}\right) - \xi^{\frac{1}{2}}\sqrt{(x^2 - 1)}\,\mathrm{F}\left(\xi, \frac{\pi}{2}\right)\right]$$
$$\mathrm{H}_{\frac{3}{2}}(x) = -\tfrac{2}{3}\left[\xi^{-\frac{1}{2}}\,x\,\mathrm{E}\left(\xi, \frac{\pi}{2}\right) - \xi^{-\frac{1}{2}}\sqrt{(x^2 - 1)}\,\mathrm{F}\left(\xi, \frac{\pi}{2}\right)\right] \quad . \quad . \quad . \qquad (48).$$

The expressions (48) give very little new information about the functions H, since we have already obtained in (32) the expression for $\mathrm{H}_n(x)$ in an ascending series of powers of ξ; but (47) throws some light on the difference between the forms (10) and (11); for having regard to the expressions (CAYLEY'S 'Elliptic Functions,' Chapter III., § 77),

$$\mathrm{F}\left(\sqrt{1 - \xi^2}, \frac{\pi}{2}\right) = \log\frac{4}{\xi} + \frac{1^2}{2^2}\xi^2\left[\log\frac{4}{\xi} - \frac{2}{1.2}\right] + \frac{1^2.3^2}{2^2.4^2}\xi^4\left[\log\frac{4}{\xi} - \frac{2}{1.2} - \frac{2}{3.4}\right] + \&\text{c.},$$
$$\mathrm{E}\left(\sqrt{1 - \xi^2}, \frac{\pi}{2}\right) = 1 + \tfrac{1}{2}\xi^2\left[\log\frac{4}{\xi} - \frac{1}{1.2}\right] + \frac{1^2.3}{2^2.4}\xi^4\left[\log\frac{4}{\xi} - \frac{2}{1.2} - \frac{1}{3.4}\right],$$

it is at once obvious that (11), which leads to (47) and (47A), gives for $\mathrm{I}_n(x)$ a function of ξ, which, in general, is not identical with (10).

We shall have to deal with infinite series into which I's and H's enter; it is, therefore, requisite to know to what values these approximate when their order, n, becomes infinite. We have, whether n is integral or fractional,

$$\mathrm{H}_n(x) = -\frac{\Pi(n-2)\,\Pi(\frac{1}{2})}{\Pi(n - \frac{1}{2})}\,\xi^{n-1}\,\mathrm{F}(\tfrac{1}{2},\ n - 1,\ n + \tfrac{1}{2},\ \xi^2) \quad . \quad . \quad . \qquad (32),$$

whence

$$\mathrm{H}_n(x) = -\tfrac{1}{2}\frac{\pi^{\frac{1}{2}}}{n^{\frac{3}{2}}}\,.\,\xi^{n-1}(1 - \xi^2)^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (49),$$

when n is very great. [HEINE, § 40.]

Again,

$$\mathrm{I}_n(x) = \frac{\xi^{-n}}{\pi}\int_{\xi^2}^1 u^n(1 - u)^{\frac{1}{2}}(u - \xi^2)^{\frac{1}{2}}\,du \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (11).$$

Now, when n is very great, each element $u^n (1-u)^{\frac{1}{2}} (u-\xi^2)^{\frac{1}{2}} du$ of the integral is very small, except when u is nearly equal to 1; we may, therefore, put

$$\mathrm{I}_n(x) = \frac{\xi^{-n}(1-\xi^2)^{\frac{1}{2}}}{\pi} \int_0^1 u^n (1-u)^{\frac{1}{2}} du$$

when n is very great,

$$= \frac{1}{\pi} \frac{\Pi(n)\,\Pi(\frac{1}{2})}{\Pi(n+\frac{3}{2})} \xi^{-n}(1-\xi^2)^{\frac{1}{2}} = \frac{1}{2\sqrt{(\pi n^3)}} \xi^{-n}(1-\xi^2)^{\frac{1}{2}} \quad . \quad . \quad . \quad (49\text{A}).$$

The expressions (49) and (49A) enable us to discuss the convergency of the development

$$\frac{1-x^2}{y-x} = \sum_{n=2}^{n=\infty} n(n-1)(2n-1)\,\mathrm{H}_n(y)\,\mathrm{I}_n(x) \quad . \quad . \quad . \quad . \quad . \quad (37).$$

For, from the equations (16) and (38), we find

$$\sum_2^n n(n-1)(2n-1)(y-x)\,\mathrm{I}_n(x)\,\mathrm{H}_n(y)$$
$$= 1 - x^2 + (n-1)\,n\,(n+1)\,[\mathrm{I}_n(x)\,\mathrm{H}_{n+1}(y) - \mathrm{H}_n(y)\,\mathrm{I}_{n+1}(x)].$$

Now the limit when n is very great of the expression in [ ] is

$$\frac{1}{4n^3}\left(\frac{\eta}{\xi}\right)^n \sqrt{(1-\xi^2 \,.\, 1-\eta^2)} \,.\, \left(1 - \frac{1}{\xi\eta}\right).$$

Hence (37) is true so long as mod. $\eta >$ mod. ξ. [HEINE, § 45.]

CHAPTER IV.—ON FUNCTIONS APPROPRIATE TO THE SPHEROID AND THE TORE.

We have seen in the introduction that the expression

$$\psi = (\mathrm{A}r^n + \mathrm{B}r^{-n+1})\,\{\mathrm{C}\,\mathrm{I}_n(\cos\theta) + \mathrm{D}\,\mathrm{H}_n(\cos\theta)\}$$

where A, B, C, D, and n are arbitrary, is a solution of

$$\mathrm{D}\psi = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3),$$

and we shall find that solutions expressed in this manner are appropriate to the discussion of questions of fluid motion, when the boundary is a sphere.

In the present chapter I propose to develop solutions which shall be appropriate to the spheroid and the tore.

Consider a spheroid whose centre is at the origin, and whose major axis, and axis of symmetry coincide in the axis of z. Let the distance between the foci on the axis of z be $2h$. Then if ph, qh, be the major axes of the confocals to this spheroid which pass through the point (ϖ, z), p^2, q^2 are the roots of the equation in λ,

$$\frac{\varpi^2}{\lambda - 1} + \frac{z^2}{\lambda} = h^2.$$

So that

$$\left.\begin{aligned} z &= hpq \\ \varpi &= ih\sqrt{(1 - p^2 \,.\, 1 - q^2)} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (50),$$

where we shall suppose p^2 algebraically greater than q^2, and, by an arbitrary convention, q to change sign with z.

From (50) we find

$$d\varpi^2 + dz^2 = h^2 (q^2 - p^2) \left(\frac{dp^2}{1 - p^2} - \frac{dq^2}{1 - q^2}\right);$$

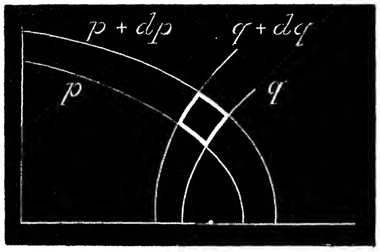
whence the normal distance between the two spheroids p, $p + dp$, is

$$h \sqrt{\left(\frac{q^2 - p^2}{1 - p^2}\right)} dp,$$

and between the hyperboloids q, $q + dq$,

$$h \sqrt{\left(\frac{p^2 - q^2}{1 - q^2}\right)} dq.$$

Now, reverting to the hydrodynamical origin of the subject, it is shown in the introduction that $-\varpi^{-1} D\psi$ is the circulation per unit area for any closed curve drawn in a meridional plane. Consider the small curve bounded by p, $p + dp$, q, $q + dq$.



The velocities in the directions of the outward-drawn normals are to p,

$$\left.\begin{aligned} \mathrm{P} &= \frac{1}{h\varpi}\sqrt{\frac{1-q^2}{p^2-q^2}}\frac{d\psi}{dq} \\ \text{to } q, \qquad \mathrm{Q} &= -\frac{1}{h\varpi}\sqrt{\frac{1-p^2}{q^2-p^2}}\frac{d\psi}{dp} \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (51),$$

and the circulation in the circuit considered is

$$+\frac{d}{dp}\left[\mathrm{Q}h\sqrt{\frac{p^2-q^2}{1-q^2}}\right]dp\,dq,$$

$$-\frac{d}{dq}\left[\mathrm{P}h\sqrt{\frac{q^2-p^2}{1-p^2}}\right]dp\,dq,$$

$$= -\left[\frac{1}{1-q^2}\frac{d^2\psi}{dp^2} - \frac{1}{1-p^2}\frac{d^2\psi}{dq^2}\right]\frac{dp\,dq}{h}$$

But this is equal to

$$-\frac{1}{\varpi}\,\mathrm{D}\psi.h\sqrt{\frac{q^2-p^2}{1-p^2}}\,dp.h\sqrt{\frac{p^2-q^2}{1-q^2}}\,dq;$$

hence

$$\mathrm{D}\psi = -\frac{1}{h^2(p^2-q^2)}\left[(1-p^2)\frac{d^2\psi}{dp^2} - (1-q^2)\frac{d^2\psi}{dq^2}\right] \quad . \; . \; . \; . \quad (6a),$$

and the equation

$$\mathrm{D}\psi = 0 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (3)$$

transforms into

$$(1-p^2)\frac{d^2\psi}{dp^2} - (1-q^2)\frac{d^2\psi}{dq^2} = 0 \quad . \; . \; . \; . \; . \; . \; . \; . \quad (3b).$$

In obtaining the above I have supposed the ellipsoids and hyperboloids of revolution about their transverse axes; and, in this case, p, q are the reciprocals of the eccentricities of the generating curves. Hence, for *ovary spheroid,*

p lies between 1 on axis, ∞ at infinity;
q ,, 1 ,, , 0 at plane of symmetry.

If we wish our equations to apply to planetary spheroids and hyperboloids of one sheet, we must suppose $-h^2$ to be the square of the radius of the focal circle in the plane of symmetry, and p, q are equal to $(e^2-1)^{\frac{1}{2}}/e$, where e is the eccentricity of the generating ellipse or hyperbola.

Hence, for *planetary spheroid,*

p lies between 0 within ring of foci, $i\infty$ at infinity, being everywhere a pure imaginary,
q lies between 0 on plane of symmetry beyond focal circle, 1 on axis.

Hence, whether we divide up space by means of ovary spheroids and hyperboloids of two sheets, or by means of planetary spheroids and hyperboloids of one sheet, q is real and lies between 0 and 1.

Hence, by p. 462, any function of the coordinates, which remains everywhere finite, may be expressed by the series $\sum\limits_{n=0}^{n=\infty} \mathrm{I}_n(q) f_n(p)$ where n is a positive integer. If this is a solution of (3) we find on substituting

$$\sum_{n=0}^{n=\infty} \mathrm{I}_n(q) \,.\, \left[(1-p^2)\frac{d^2 f_n(p)}{dp^2} + n(n-1) f_n(p)\right] = 0,$$

whence,

$$(1-p^2)\frac{d^2 f_n(p)}{dp^2} + n(n-1) f_n(p) = 0,$$

which is the equation (7).

Hence, any solution of (3) may be written

$$\Sigma\, \mathrm{I}_n(q)\,[\mathrm{A}_n \mathrm{I}_n(p) + \mathrm{B}_n \mathrm{H}_n(p)]$$

where n is a positive integer.

When n is a positive integer $\mathrm{I}_n(p)\,\mathrm{I}_n(q)$ is a rational integral function of the coordinates ϖ, z, of degree n.

In the first place, let n be even. The product may be resolved into a series of products, $(p^2-\alpha^2)(q^2-\alpha^2)$ where α is any root of $\mathrm{I}_n(x) = 0$.

And by (50),

$$(p^2-\alpha^2)(q^2-\alpha^2) = -\frac{\varpi^2}{h^2},$$

when $\alpha = 1$;

$$= \frac{\alpha^2(1-\alpha^2)}{h^2}\left[\frac{z^2}{\alpha^2} + \frac{\varpi^2}{1-\alpha^2} - h^2\right],$$

when $\alpha \neq 1$.

But by (9) we see that, if $\Pi(\alpha^2)$ denotes the product of all the α^2 s,

$$\Pi(\alpha^2) = \frac{1.2\ldots n}{2.4\ldots n.n-1.n+1\ldots 2n-3},$$

and

$$\Pi(1-\alpha^2) = \operatorname*{Lt}_{x=1} \frac{\mathrm{I}_n(x)}{x^2-1} = \frac{\mathrm{P}_{n-1}(1)}{2} = \tfrac{1}{2}.$$

Hence, when n is even,

$$\mathrm{I}_n(p)\,\mathrm{I}_n(q) = -\frac{1.2\ldots n}{2.4\ldots n.n-1\ldots 2n-3}\cdot\frac{\varpi^2}{2h^{2n}}\,\Pi\left(\frac{z^2}{\alpha^2} + \frac{\varpi^2}{1-\alpha^2} \quad h^2\right) \quad . \quad . \quad (52),$$

where all the roots occur in the product excepting $\alpha^2 = 1$.

Similarly, when n is odd,

$$\mathrm{I}_n(p)\,\mathrm{I}_n(q) = -\frac{1.2\ldots n}{2.4\ldots n-1.n.n+2\ldots 2n-3}\cdot\frac{z\varpi^2}{2h^{2n+1}}\,\Pi\left(\frac{z^2}{\alpha^2} + \frac{\varpi^2}{1-\alpha^2} - h^2\right),$$

where, under the product, all the roots are included, excepting $\alpha = \pm 1$ and $\alpha = 0$, the latter of which gives the term pq, or z/h.

Now it is easy to see that, except for a constant factor, there is only one homo geneous rational function of the coordinates of positive degree r which satisfies (3). For such a function in its most general form involves $r+1$ constants, and, when we have operated upon it with $\varpi\left(\frac{d^2}{d\varpi^2}+\frac{d^2}{dz^2}\right)-\frac{d}{d\varpi}$, we are left with a similar function of degree $r-1$, each of whose terms must vanish. This gives r linear equations between the $r+1$ constants, showing that one, and one only, is independent, and that appears as an arbitrary factor.

Hence the r functions $\mathrm{I}_r(p)\,\mathrm{I}_r(q)$, $\mathrm{I}_{r-1}(p)\,\mathrm{I}_{r-1}(q), \ldots$ must be equivalent to these r homogeneous solutions of (3), and are therefore sufficient for expressing any rational integral function of the coordinates of positive degree r, which satisfies (3).

We shall require hereafter the solution of

$$(1-p^2)\frac{d^2\psi}{dp^2}-(1-q^2)\frac{d^2\psi}{dq^2}=f_m(p)\,\phi_n(q) \quad . \quad . \quad . \quad . \quad . \quad (3c),$$

where $f_m(p)$, $\phi_n(q)$ are linear functions of $\mathrm{I}_m(p)$ and $\mathrm{H}_m(p)$, $\mathrm{I}_n(q)$ and $\mathrm{H}_n(q)$, and $m \neq n$.

A particular integral is

$$\psi=\frac{f_m(p)\;\phi_n(q)}{(n-m)(n+m-1)}.$$

I shall now consider the functions which are appropriate to the tore.

Write

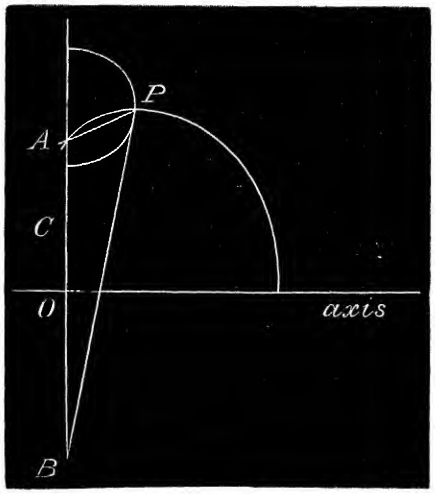
$$\varpi+iz=c\coth\frac{\eta-i\xi}{2},$$

$$=c\,\frac{\sinh\eta+i\sin\xi}{\cosh\eta-\cos\xi} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (53).$$

Then

$$\varpi^2+z^2-2zc\cot\xi-c^2=0,$$

$$\varpi^2+z^2-2\varpi c\coth\eta+c^2=0.$$



And if P be any point in a meridional plane, A, B two points in that plane symmetrically placed at distance c on opposite sides of the axis of symmetry,

$$e^{\eta} = \frac{\mathrm{BP}}{\mathrm{AP}},$$

$$\xi = \mathrm{APB}.$$

Hence the surfaces represented by $\eta =$ const. are tores, and those represented by $\xi =$ const., the surfaces given by the intersection of two equal spheres.

We also find

$$d\varpi^2 + dz^2 = c^2 \frac{d\eta^2 + d\xi^2}{(\cosh\eta - \cos\xi)^2}.$$

Hence the normal distance between the surfaces η, $\eta + d\eta$ is

$$\frac{c\,d\eta}{\cosh\eta - \cos\xi},$$

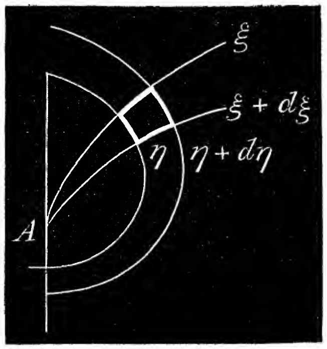
and between ξ, $\xi + d\xi$,

$$\frac{c\,d\xi}{\cosh\eta - \cos\xi}.$$

Thus the velocities in the directions of the outward-drawn normals to the surfaces η, ξ are

$$\left.\begin{aligned} \mathrm{H} &= -\frac{\cosh\eta - \cos\xi}{c\varpi}\frac{d\psi}{d\xi} \\ \Xi &= -\frac{\cosh\eta - \cos\xi}{c\varpi}\frac{d\psi}{d\eta} \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (54),$$

and hence the circulation in the circuit formed by η, $\eta + d\eta$, ξ, $\xi + d\xi$,



$$-\frac{d}{d\xi}\left[\mathrm{H}\frac{c}{\cosh\eta - \cos\xi}\right] d\xi\,d\eta + \frac{d}{d\eta}\left[-\Xi\frac{c}{\cosh\eta - \cos\xi}\right] d\xi\,d\eta,$$

$$= \left\{\frac{d}{d\xi}\left[\frac{1}{\varpi}\frac{d\psi}{d\xi}\right] + \frac{d}{d\eta}\left[\frac{1}{\varpi}\frac{d\psi}{d\eta}\right]\right\} d\xi\,d\eta,$$

and we get

$$\mathrm{D}\psi = -(\cosh\eta - \cos\xi)^2\varpi\left[\frac{d}{d\xi}\left(\frac{1}{\varpi}\frac{d\psi}{d\xi}\right) + \frac{d}{d\eta}\left(\frac{1}{\varpi}\frac{d\psi}{d\eta}\right)\right] \quad . \; . \; . \quad (6b)$$

where

$$\varpi = \frac{c\sinh\eta}{\cosh\eta - \cos\xi}.$$

If we write

$$\psi = \chi/(\cosh\eta - \cos\xi)^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (55),$$

$$\left.\begin{aligned} \frac{1}{\varpi}\frac{d\psi}{d\xi} &= \frac{1}{c\sinh\eta}\left[(\cosh\eta - \cos\xi)^{\frac{1}{2}}\frac{d\chi}{d\xi} - \frac{\chi\sin\xi}{2(\cosh\eta - \cos\xi)^{\frac{1}{2}}}\right] \\ \frac{1}{\varpi}\frac{d\psi}{d\eta} &= \frac{1}{c\sinh\eta}\left[(\cosh\eta - \cos\xi)^{\frac{1}{2}}\frac{d\chi}{d\eta} - \frac{\chi\sinh\eta}{2(\cosh\eta - \cos\xi)^{\frac{1}{2}}}\right] \end{aligned}\right\} \quad . \quad . \quad (56),$$

and we find

$$\mathrm{D}\psi = -\frac{(\cosh\eta - \cos\xi)^{\frac{3}{2}}}{c^2}\left[\frac{d^2\chi}{d\xi^2} + \frac{d^2\chi}{d\eta^2} - \coth\eta\frac{d\chi}{d\eta} + \tfrac{1}{4}\right],$$

$$= -\frac{(x - \cos\xi)^{\frac{3}{2}}}{c^2}\left[\frac{d^2}{d\xi^2} - (1 - x^2)\frac{d^2}{dx^2} + \tfrac{1}{4}\right](x - \cos\xi)^{\frac{1}{2}} . \psi \quad . \quad . \quad (6c),$$

where

$$x = \cosh\eta,$$

and

$$\mathrm{D}\psi = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

transforms into

$$\frac{d^2\psi}{d\xi^2} - (1 - x^2)\frac{d^2\chi}{dx^2} + \frac{\chi}{4} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3c),$$

where

$$\chi = \psi(\cosh\eta - \cos\xi)^{\frac{1}{2}}.$$

Now ξ, as it occurs in the equation (3c), is a real angle lying between π and 0. Hence, any function, χ, which satisfies (3c), may be expressed, by FOURIER'S theorem, in a series

$$\Sigma f_n(x)\cos(n\xi + \alpha_n),$$

where n is a positive integer.

Substitute in (3c), and we find

$$\Sigma - \cos(n\xi + \alpha_n)\left[(1 - x^2)\frac{d^2 f_n(x)}{dx^2} + (n^2 - \tfrac{1}{4})f_n(x)\right] = 0,$$

whence

$$(1 - x^2)\frac{d^2 f_n(x)}{dx^2} + (n + \tfrac{1}{2})(n - \tfrac{1}{2})f_n(x) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (7),$$

which shows that

$$f_n(x) = \mathrm{A}_n\mathrm{I}_{n+\frac{1}{2}}(x) + \mathrm{B}_n\mathrm{H}_{n+\frac{1}{2}}(x),$$

n being a positive integer, and any solution of (3) may be written

$$\psi = \frac{1}{(x - \cos\xi)^{\frac{1}{2}}}\Sigma\cos(n\xi + \alpha_n)\left[\mathrm{A}_n\mathrm{I}_{n+\frac{1}{2}}(x) + \mathrm{B}_n\mathrm{H}_{n+\frac{1}{2}}(x)\right] \quad . \quad . \quad . \quad (57),$$

an expression identical, except as regards notation, with that given by Mr. HICKS. ('Phil. Trans.,' 1884.)

We are now in a position to show how the expression given on p. 473 for $\mathrm{H}_{n+\frac{1}{2}}(x)$ may be derived. [*Cf.* HICKS, 'Phil. Trans.,' 1884.]

For $\psi = \sqrt{(\varpi^2 + z^2)}$ is a solution of (3) ; and

$$\varpi^2 + z^2 = \frac{\cosh \eta + \cos \xi}{\cosh \eta - \cos \xi}$$

Hence we may write

$$(x + \cos \xi)^{\frac{1}{2}} = \sum_{n=0}^{n=\infty} \cos (n\xi + \alpha_n) \left[\mathrm{A}_n \mathrm{I}_{n+\frac{1}{2}}(x) + \mathrm{B}_n \mathrm{H}_{n+\frac{1}{2}}(x) \right].$$

But it is clear that the left-hand member may be developed in cosines of multiples of ξ, and, therefore, $\alpha_n = 0$.

Hence

$$\mathrm{A}_n \mathrm{I}_{n+\frac{1}{2}}(x) + \mathrm{B}_n \mathrm{H}_{n+\frac{1}{2}}(x) = \frac{2}{\pi} \int_0^\pi \cos n\xi \sqrt{(x + \cos\xi)}\, d\xi.$$

Now, when $x = \infty$, the right-hand member vanishes. In fact, since

$$\int_0^\pi \cos n\xi \cos^r \xi \, d\xi = 0,$$

when n, r are positive integers and $r < n$, it is clear that the highest power occurring on the right when the expression is developed in descending powers of x, is $x^{-n+\frac{1}{2}}$. Hence $\mathrm{A}_n = 0$. To find B_n, put $x = 1$.

$$\int_0^\pi \cos n\xi \sqrt{2} \cos \frac{\xi}{2} \, d\xi = \frac{(-1)^{n-1}}{\sqrt{2}} \frac{1}{(n + \frac{1}{2})(n - \frac{1}{2})}.$$

But

$$\mathrm{H}_{n+\frac{1}{2}}(1) = - \frac{1}{(n + \frac{1}{2})(n - \frac{1}{2})},$$

therefore

$$\mathrm{B}_n = \frac{(-1)^n}{\pi}$$

and

$$\mathrm{H}_{n+\frac{1}{2}}(x) = (-1)^n \int_0^\pi \cos n\xi \sqrt{\{2(x + \cos \xi)\}}\, d\xi$$

$$= \int_0^\pi \cos n\xi \sqrt{\{2(x - \cos \xi)\}}\, d\xi \quad \ldots\ldots \quad (46),$$

where n is a positive integer, and x is real and > 1.

These functions have been employed by Mr. HICKS in the papers already referred to ; and, except for purely mathematical interest, it is chiefly in connection with the theory of fluid-motion about a tore that functions of fractional order claim our notice.

Mr. BASSET* has, however, employed the function $\mathrm{L}_n = \frac{2}{\pi}(-1)^n e^{-n/2} \mathrm{H}_{n+\frac{1}{2}}(\cosh \eta)$, to find the current function for motion in an infinite liquid due to the rotation of an infinite cylinder whose cross-section is a lemniscate.

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* 'Hydrodynamics,' vol. 1, § 115.

Any solution of (3), which consists exclusively of positive integral powers of the coordinates ϖ, z, can be written in the form

$$\frac{1}{(x-\cos\xi)^{\frac{1}{2}}}\sum_{n=o}^{n=\infty}\mathrm{B}_n\mathrm{H}_{n+\frac{1}{2}}(x)\cos(n\xi+\alpha_n).$$

To arrive at this theorem we notice that

$$2\frac{d}{d\xi}(x-\cos\xi)^{\frac{1}{2}}=z(x-\cos\xi)^{\frac{1}{2}}.$$

But

$$2\frac{d}{d\xi}=-\varpi z\frac{d}{d\varpi}+(\varpi^2-z^2-1)\frac{d}{dz};$$

hence we see that

$$2^r\left(\frac{d}{d\xi}\right)^r(x-\cos\xi)^{\frac{1}{2}}=\mathrm{U}_r(x-\cos\xi)^{\frac{1}{2}},$$

where U_r is a rational integral function of ϖ, z, of degree r, and consecutive functions U_r, U_{r+1} are connected by the relation

$$\mathrm{U}_{r+1}=-2\varpi z\frac{d\mathrm{U}_r}{d\varpi}+(\varpi^2-z^2-1)\frac{d\mathrm{U}_r}{dz}+z\mathrm{U}_r,$$

and $\mathrm{U}_0=1$.

Forming the first five functions, we have

$$\mathrm{U}_1=z,$$
$$\mathrm{U}_2=\varpi^2-1,$$
$$\mathrm{U}_3=z(-3\varpi^2-1),$$
$$\mathrm{U}_4=-3\varpi^4+12\varpi^2z^2+2\varpi^2+1,$$
$$\mathrm{U}_5=z(45\varpi^4-60\varpi^2z^2-30\varpi^2-1).$$

From these it is clear that when r is odd, z is a factor of U_r; but that, otherwise, U_r is an even function of the coordinates, ϖ, z, $-\varpi$ occurring in higher power than z.

Now

$$(x-\cos\xi)^{\frac{1}{2}}=\sum_{n=0}^{n=\infty}\frac{1}{\pi}\mathrm{H}_{n+\frac{1}{2}}(x)\cos n\xi.$$

Therefore,

$$\mathrm{U}_r=\frac{1}{(x-\cos\xi)^{\frac{1}{2}}}\sum_{n=0}^{n=\infty}\cdot\frac{(2n)^r}{\pi}\mathrm{H}_{n+\frac{1}{2}}(x)\cos\left(n\xi+\frac{r\pi}{2}\right),$$

and therefore $\psi=\mathrm{U}_r$ is a solution of (3). But there are only r rational functions of the coordinates of positive integral degree not greater than r, which satisfy (3), and which are independent. Hence, in terms of the functions, U_r, U_{r-1}, &c., we can express any such solution of (3); whence we obtain the theorem stated at the beginning of this section, and we see that B_n is a rational integral function of n, of the same degree in n as the solution of (3) is in the coordinates, ϖ, z.

TABLE I.—Functions of ϖ and z of Positive Degree satisfying

$$\frac{d^2\psi}{d\varpi^2} + \frac{d^2\psi}{dz^2} - \frac{1}{\varpi}\frac{d\psi}{d\varpi} = 0.$$

(1)	$z.$
(2)	$\varpi^2.$
(3)	$z\varpi^2.$
(4)	$\varpi^2(4z^2 - \varpi^2).$
(5)	$z\varpi^2(4z^2 - 3\varpi^2).$
(6)	$\varpi^2(8z^4 - 12\varpi^2 z^2 + \varpi^4).$
(7)	$z\varpi^2(8z^4 - 20z^2\varpi^2 + 5\varpi^4).$
(8)	$\varpi^2(64z^6 - 240z^4\varpi^2 + 120z^2\varpi^4 - 5\varpi^6).$
(9)	$z\varpi^2(64z^6 - 336z^4\varpi^2 + 280z^2\varpi^4 - 35\varpi^6).$
(10)	$\varpi^2(128z^8 - 896z^6\varpi^2 + 1120z^4\varpi^4 - 280z^2\varpi^6 + 7\varpi^8).$

The above were calculated from the formula

$$U_n = (2n - 3) zU_{n-1} - (\varpi^2 + z^2)\frac{dU_{n-1}}{dz},$$

which may be readily deduced from (13).

TABLE II.—$I_n(x)$ in powers of x.

$n = 0.$	$-1.$
1.	$x.$
2.	$\dfrac{x^2 - 1}{2}.$
3.	$\dfrac{x^3 - x}{2}.$
4.	$\dfrac{5x^4 - 6x^2 + 1}{8}.$
5.	$\dfrac{7x^5 - 10x^3 + 3x}{8}$
6.	$\dfrac{21x^6 - 35x^4 + 15x^2 - 1}{16}.$
7	$\dfrac{33x^7 - 63x^5 + 35x^3 - 5x}{16}.$
8	$\dfrac{429x^8 - 924x^6 + 630x^4 - 140x^2 + 5}{128}.$
9.	$\dfrac{715x^9 - 1716x^7 + 1386x^5 - 420x^3 + 35x}{128}.$
10.	$\dfrac{2431x^{10} - 6435x^8 + 6006x^6 - 2310x^4 + 315x^2 - 7}{256}.$

TABLE III.—$K_n(x)$ in powers of x.

$n = 2;\quad \dfrac{x}{2}.$

$3;\quad \dfrac{x^2}{2} - \frac{1}{3}.$

$4;\quad \frac{5}{8}x^3 - \frac{13}{24}x.$

$5;\quad \frac{7}{8}x^4 - \frac{23}{24}x^2 + \frac{2}{15}.$

$6;\quad \frac{21}{16}x^5 - \frac{7}{4}x^3 + \frac{113}{240}x.$

$7;\quad \frac{33}{16}x^6 - \frac{13}{4}x^4 + \frac{103}{80}x^2 - \frac{8}{105}.$

$8;\quad \frac{429}{128}x^7 - \frac{781}{128}x^5 + \frac{2039}{640}x^3 - \frac{1873}{4480}x.$

$9;\quad \frac{715}{128}x^8 - \frac{4433}{384}x^6 + \frac{957}{128}x^4 - \frac{6967}{4480}x^2 + \frac{16}{315}.$

$10;\quad \frac{2431}{256}x^9 - \frac{8437}{384}x^7 + \frac{2717}{160}x^5 - \frac{4367}{896}x^3 + \frac{30563}{80460}x.$

These were calculated from the sequence equations (38), (38A).

$$nK_n(x) = (2n-3)\,xK_{n-1}(x) - (n-3)\,K_{n-2}(x),$$
$$3K_3(x) = 3xK_2(x) - 1.$$

TABLE IV.—x^n in terms of $I(x)$'s.

$n = 0;\quad -I_0.$

$1;\quad I_1.$

$2;\quad 2I_2 - I_0.$

$3;\quad 2I_3 + I_1.$

$4;\quad \frac{8}{5}I_4 + \frac{12}{5}I_2 - I_0.$

$5;\quad \frac{8}{7}I_5 + \frac{20}{7}I_3 + I_1,$

$6;\quad \frac{16}{21}I_6 + \frac{8}{3}I_4 + \frac{18}{7}I_2 - I_0.$

$7;\quad \frac{16}{33}I_7 + \frac{24}{11}I_5 + \frac{10}{3}I_3 + I_1.$

$8;\quad \frac{128}{429}I_8 + \frac{64}{39}I_6 + \frac{112}{33}I_4 + \frac{8}{3}I_2 - I_0.$

$9;\quad \frac{128}{715}I_9 + \frac{64}{55}I_7 + \frac{432}{143}I_5 + \frac{40}{11}I_3 + I_1.$

$10;\quad \frac{256}{2431}I_{10} + \frac{1920}{2431}I_8 + \frac{32}{13}I_6 + \frac{560}{143}I_4 + \frac{30}{11}I_2 - I_0.$

TABLE V.—Cos $(n\cos^{-1}x)$ in terms of $I(x)$'s.

$n = 0;\quad -I_0.$

$1;\quad I_1.$

$2;\quad 4I_2 - I_0.$

$3;\quad 8I_3 + I_1.$

$4;\quad \frac{64}{5}I_4 + \frac{16}{5}I_2 - I_0.$

$5;\quad \frac{128}{7}I_5 + \frac{40}{7}I_3 + I_1.$

$6;\quad \frac{512}{21}I_6 + \frac{128}{15}I_4 + \frac{108}{35}I_2 - I_0.$

$7;\quad \frac{1024}{33}I_7 + \frac{128}{11}I_5 + \frac{16}{3}I_3 + I_1.$

$8;\quad \frac{16384}{429}I_8 + \frac{4096}{273}I_6 + \frac{256}{33}I_4 + \frac{64}{21}I_2 - I_0.$

$9;\quad \frac{32768}{715}I_9 + \frac{1024}{55}I_7 + \frac{10368}{1001}I_5 + \frac{400}{77}I_3 + I_1.$

$10;\quad \frac{131072}{2431}I_{10} + \frac{163840}{7293}I_8 + \frac{512}{39}I_6 + \frac{3200}{429}I_4 + \frac{100}{33}I_2 - I_0.$

TABLE VI.—Various Forms of $\mathrm{I}_n(x)$.

$$2^{n-1}\frac{\Pi\left(n-\frac{3}{2}\right)}{\Pi(n)\,\Pi\left(-\frac{1}{2}\right)}\,x^n\,\mathrm{F}\left(-\frac{n}{2},\,-\frac{n}{2}+\tfrac{1}{2},\,-n+\tfrac{3}{2},\,x^{-2}\right) \tag{9}$$

$$(-1)^{n/2}\,\frac{\Pi\left(\dfrac{n}{2}-\frac{3}{2}\right)}{2\Pi\left(\dfrac{n}{2}\right)\Pi\left(-\frac{1}{2}\right)}\,\mathrm{F}\left(-\frac{n}{2},\,\frac{n}{2}-\tfrac{1}{2},\,\tfrac{1}{2},\,x^2\right) \tag{9A}$$

$$(-1)^{\frac{1}{2}(n-1)}\,\frac{\Pi\left(\dfrac{n}{2}-1\right)}{\Pi\left(\dfrac{n}{2}-\frac{1}{2}\right)\Pi\left(-\frac{1}{2}\right)}\,x\mathrm{F}\left(-\frac{n}{2}+\tfrac{1}{2},\,\frac{n}{2},\,\tfrac{3}{2},\,x^2\right) \tag{9B}$$

$$\frac{\Pi\left(n-\frac{3}{2}\right)}{\Pi(n)\,\Pi\left(-\frac{1}{2}\right)}\cdot\left[\cos\left(n\cos^{-1}x\right)-\frac{2n}{2\,(2n-3)}\cos\left(\overline{n-2}\,\cos^{-1}x\right)-\ldots\right] \tag{10A}$$

$$\frac{\Pi\left(n-\frac{3}{2}\right)}{2\Pi(n)\,\Pi\left(-\frac{1}{2}\right)}\,\xi^{-n}\,\mathrm{F}\left(-\tfrac{1}{2},\,-n,\,-n+\tfrac{3}{2},\,\xi^2\right) \tag{10}$$

$$\frac{\xi^{-n}}{\pi}\int_{\xi^2}^{1}u^{n-2}\,(1-u)^{\frac{1}{2}}\,(u-\xi^2)^{\frac{1}{2}}\,du \tag{11}$$

$$\frac{1}{\underline{|n-1}}\left(\frac{d}{dx}\right)^{n-2}\left(\frac{x^2-1}{2}\right)^{n-1} \tag{12}$$

$$(-1)^{n-1}\,\frac{r^{n-1}}{\underline{|n}}\,\frac{d^n r}{dz^n} \tag{13}$$

$$\frac{1}{\pi}\int_0^{\pi}\sin^2\eta\,(x^2-1)\,\left(x-\cos\eta\,\sqrt{x^2-1}\right)^{n-2}d\eta \tag{14}$$

$$\frac{1}{\pi}\int_0^{\pi}\sin^2\phi\,(x^2-1)\left(x+\cos\phi\,\sqrt{x^2-1}\right)^{-n-1}d\phi \tag{14A}$$

$$-\frac{1}{\pi}\int_{-\cosh^{-1}x}^{\cosh^{-1}x}d\phi\,\sqrt{\{2\,(x-\cosh\phi)\}}\;e^{\left(n-\frac{1}{2}\right)\phi} \tag{15}$$

$$-\frac{2}{\pi}\int_0^{\cos^{-1}x}d\phi\,\sqrt{\{2\,(\cos\phi-x)\}}\;\cos\left(n-\tfrac{1}{2}\right)\phi \tag{15B}$$

$$\frac{2}{\pi}\int_{\cos^{-1}x}^{\pi}d\phi\,\sqrt{\{2\,(x-\cos\phi)\}}\;\sin\left(n-\tfrac{1}{2}\right)\phi \tag{15C}$$

TABLE VII.—Various Forms of $H_n(x)$.

$$-\,2^{-n}\frac{\Pi\,(n-2)\,\Pi\,(-\frac{1}{2})}{\Pi\,(n-\frac{1}{2})}\,x^{-n+1}\,\mathrm{F}\left(\frac{n-1}{2},\frac{n}{2},n-\tfrac{1}{2},x^{-2}\right)\quad .\;.\quad (28)$$

$$(-1)^{n/2}\,\frac{\Pi\left(\dfrac{n}{2}-1\right)\Pi\,(\frac{1}{2})}{\Pi\left(\dfrac{n}{2}-\frac{1}{2}\right)}\,x\,\mathrm{F}\left(-\frac{n}{2}+\tfrac{1}{2},\frac{n}{2},\tfrac{3}{2},x^2\right)\quad .\;.\;.\;.\quad (28\text{A})$$

$$(-1)^{\frac{1}{2}(n+1)}\,\frac{\Pi\left(\dfrac{n}{2}-\frac{3}{2}\right)\Pi\,(\frac{1}{2})}{\Pi\left(\dfrac{n}{2}\right)}\,\mathrm{F}\left(-\frac{n}{2},\frac{n}{2}-\tfrac{1}{2},\tfrac{1}{2},x^2\right)\quad .\;.\quad (28\text{B})$$

$$-\,\frac{\Pi\,(n-2)\,\Pi\,(\frac{1}{2})}{\Pi\,(n-\frac{1}{2})}\,\xi^{n-1}\,\mathrm{F}\,(-\tfrac{1}{2},n-1,n+\tfrac{1}{2},\xi^2)\quad .\;.\;.\quad (32)$$

$$-\,\xi^{n-1}\int_0^1 u^{n-2}\,(1-u)^{\frac{1}{2}}\,(1-u\xi^2)^{\frac{1}{2}}\,du\quad .\;.\;.\;.\;.\quad (33)$$

$$-\int_0^{\log\,(x+1)/(x-1)} d\phi\,(x^2-1)\sinh^2\phi\,(x-\sqrt{x^2-1}\,\cosh\phi)^{n-2}\quad .\quad (34)$$

$$-\int_0^{\infty} d\theta\,(x^2-1)\sinh^2\theta\,(x+\sqrt{x^2-1}\,\cosh\theta)^{-n-1}\quad .\;.\;.\quad (35)$$

$$-\int_{-\infty}^{\cosh^{-1}x} d\phi\,\sqrt{\{2\,(\cosh\phi-x)\}}\,e^{(n-\frac{1}{2})\,\phi}\quad .\;.\;.\;.\;.\quad (36)$$

$$\left\{\begin{array}{lr}
\frac{1}{2}\,\mathrm{I}_n(x)\log\dfrac{x+1}{x-1}-\mathrm{K}_n(x) & (44)\\[2ex]
{}^{*}\,\frac{1}{2}\,\mathrm{I}_n(x)\log\dfrac{1+x}{1-x}-\mathrm{K}_n(x) & (44\text{C})\\[2ex]
\frac{1}{2}\displaystyle\int_{-1}^{+1}\frac{\mathrm{I}_n(y)}{x-y}\,dy & (44\text{D})\\[2ex]
\displaystyle\int_0^{\pi}\cos\,(n-\tfrac{1}{2})\,\theta\,\sqrt{\{2\,(x-\cos\theta)\}}\,d\theta & (46)
\end{array}\right.$$

(9), (9A), (9B), (10), (10A), (12), (13), (15C), (28A), (28B), (44), (44C), and (44D) require n to be positive integer. (46) requires n of form *integer* $+\frac{1}{2}$. (9A) and (28A) n even; (9B) and (28B) n odd. (28A), (28B), mod. $x \ngtr 1$. (15C) and (44C) x real and < 1. (28) mod. $x \nless 1$. (46) x real and > 1. (14A) x not a pure imaginary. (35) x not real, negative, and > 1. Subject to these limitations, the above forms give the same values, with the exception of (44C) among the functions H.

CHAPTER V.—IRROTATIONAL MOTION IN HYDRODYNAMICS.

The chief interest attaching to STOKES's Current Function lies in its application where the motion is rotational, and, consequently, where the Velocity Potential fails us. But it is never without value, since the equation $\psi =$ const. gives the stream surfaces. Moreover, methods are best illustrated by their simpler applications, and the results serve for comparison with more complicated cases. Accordingly, in this chapter I find the value of ψ for certain known cases of liquid motion.

The problems we can attack with STOKES's Current Function are those of motion symmetrical with respect to an axis, and this requires that the boundaries shall be symmetrical. The boundaries may, therefore, be such surfaces as a sphere, a tore, a hyperboloid of revolution of one sheet. We may suppose the boundary and the liquid at infinity, either at rest or in motion, symmetrically. When the liquid at infinity is in motion, I shall suppose its motion uniform; and we may then reduce any of the cases within this range to the problem of flux past a fixed obstacle, of liquid moving uniformly at infinity.

The motion being irrotational, we have $D\psi = 0$, and consequently ψ consists of a series of terms such as we have found as solutions of (3) in the foregoing chapters. If these are expressed as functions of the coordinates ϖ, z, it is easy to see that the only positive power of the coordinates that can occur is ϖ^2; for any higher power would make the velocity infinite at infinity, and the term z would make it infinite at the axis. This immediately strikes out various terms. For example,

$r \cos \theta$; $r^3 \mathrm{I}_3 (\cos \theta)$; $r^4 \mathrm{I}_4 (\cos \theta)$, &c., and $\mathrm{I}_1 (p) \mathrm{I}_1 (q)$, $\mathrm{I}_3 (p) \mathrm{I}_3 (q) \ldots$ by p. 479.

Our process is, then, to assume a value for ψ which contains a complete series of terms, omitting these impossible ones. The mode of expression that is appropriate differs with the form of boundary. We then use the principle that $\psi =$ const. is a stream surface, and may, therefore, be made to represent the surface of the obstacle; while at infinity $\psi = -\frac{1}{2} \mathrm{V} \varpi^2$, where V is the velocity there from right to left.

1. *Fixed Spherical Obstacle.*

We may assume

$$\psi = \mathrm{A} + \mathrm{B} \cos \theta + \left(\mathrm{C} r^2 + \frac{\mathrm{D}}{r}\right) \mathrm{I}_2 (\cos \theta) + \frac{\mathrm{E}}{r^2} \mathrm{I}_3 (\cos \theta) +, \text{\&c.},$$

the centre being the origin.

Then, if a is the radius, $\mathrm{B} = \mathrm{E} = \ldots = 0$, and

$$\mathrm{C} a^2 + \frac{\mathrm{D}}{a} = 0.$$

But

$$r^2 \mathrm{I}_2 (\cos \theta) = -\frac{\varpi^2}{2}.$$

Hence

$$\psi = \mathrm{V}\left(r^2 - \frac{a^3}{r}\right) \mathrm{I}_2 (\cos \theta)$$

$$= -\tfrac{1}{2} \mathrm{V}\left(r^2 - \frac{a^3}{r}\right) \sin^2 \theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (58).$$

2. *Spheroidal Obstacle.*

Assume

$$\begin{aligned}\psi = \mathrm{A} &+ [\mathrm{BI}_0 (p) + \mathrm{CH}_0 (p)] \, \mathrm{I}_0 (q) \\ &+ [\mathrm{DI}_1 (p) + \mathrm{EH}_1 (p)] \, \mathrm{I}_1 (q) \\ &+ [\mathrm{FI}_2 (p) + \mathrm{GH}_2 (p)] \, \mathrm{I}_2 (q) \\ &+ [\mathrm{KI}_3 (p) + \mathrm{LH}_3 (p)] \, \mathrm{I}_3 (q) + \ldots \quad . \quad . \quad . \quad . \quad . \quad . \quad (59).\end{aligned}$$

Putting D, K, &c., zero, in accordance with p. 489, the first terms become

$$\mathrm{B} - \mathrm{C}p + \mathrm{E}q,$$

since

$$\mathrm{H}_0 (p) = p. \qquad \mathrm{H}_1 (p) = 1.$$

Of these the term Cp must be excluded; for referring to Chapter IV., p. 478, we see that the velocities normal to the surfaces p, q, are

$$\mathrm{P} = \frac{1}{h\varpi} \sqrt{\frac{1 -}{p^2 - q^2}} \frac{d\psi}{dq}$$

$$\mathrm{Q} = -\frac{1}{h\varpi} \sqrt{\frac{1 - p^2}{q^2 - p^2}} \frac{d\psi}{dp} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (51).$$

Hence the presence of this term would make Q infinite at the axis, where

$$0 = \varpi = ih \sqrt{(1 - p^2 \,.\, 1 - q^2)}.$$

These expressions, in fact, tell us that where we may have $1 - q^2 = 0$, the terms of ψ that involve p, must contain the factor $1 - q^2$, and *vice versâ*. The loci to which these values correspond are given on p. 478. They are, namely, for

Ovary spheroid and hyperboloid of two sheets:—

$p = 1$ on axis; ∞ at infinity.
$q = 1$ on axis; 0 at plane of symmetry.

Planetary spheroid and hyperboloid of one sheet:—

$p = 0$ within ring of foci; $i\,\infty$ at infinity.
$q = 1$ on axis; 0 on plane of symmetry beyond focal circle.

From these we see, too, that p is the member that becomes infinite, and hence, whether it occurs in the term $I_n(p)\,I_n(q)$ or not, or whether ψ is a solution of $D\psi = 0$ or not, no higher positive power of p than the second can occur in ψ.

Thus, in the region outside a spheroidal obstacle, we may have

$$\psi = A + Eq + [FI_2(p) + GH_2(p)]\,I_2(q) + LH_3(p)\,I_3(q) + \ldots$$

giving

$$E = L = \ldots = 0,$$

$$\psi = 2Vh^2 I_2(q)\,.\,I_2(p_0)\left[\frac{I_2(p)}{I_2(p_0)} - \frac{H_2(p)}{H_2(p_0)}\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad (60),$$

where p_0 is the parameter of the spheroidal obstacle.

When p_0 is very small, that is to say, when the obstacle approximates to a disc,

$$\psi = 2Vh^2 I_2(q)\left[I_2(p) + \frac{H_2(p)}{2p_0}\right],$$

a form which shows that the velocity becomes infinite at the edge of a disc.

A solution will be given hereafter, namely, on the supposition that the liquid is viscous, and the motion slow, which makes the velocity zero at the edge. See p. 504.

3. *Within a Hyperboloid.*

Let us consider the motion within a hyperboloid of one sheet, q_0, let us say. The velocity cannot, of course, be finite at infinity, and V will here denote its value at the centre of the hyperboloid.

We may assume

$$\psi = A + Eq + GH_2(p)\,I_2(q) + LH_3(p)\,I_3(q)\ldots,$$

and we find that all the coefficients must vanish except A and E; giving

$$\psi = Vh^2 q \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (61).$$

It is easy to see that if we attempt to proceed to the case of flux through a circular hole in a wall, we find an infinite velocity at the edge. See p. 509.

4. *The Obstacle a Tore.*

The solution in this case, and in terms of the functions which I have been discussing in this paper, has been given by Mr. HICKS ('Phil. Trans.,' 1884), who also considers the cyclic motion which can take place. No useful purpose would be served by quoting here Mr. HICKS's results. I shall, therefore, confine myself to proving rigorously a point which he assumes, namely, that in the expression of the current function due to the motion of a tore through a liquid at rest at infinity, in the form (57), the functions H do not enter.

This is deserving of some notice, since it is no longer true when the liquid at infinity is in motion.

Assuming (p. 482)

$$\psi = \frac{1}{(x - \cos\xi)^{\frac{1}{2}}} \sum_{n=0}^{n=\infty} \{A_n I_{n+\frac{1}{2}}(x) + B_n H_{n+\frac{1}{2}}(x)\} \cos(n\xi + \alpha_n) \quad . \quad . \quad (57),$$

the condition that the velocity should be continuous requires $d\psi/d\xi = 0$ at the axis, *i.e.*, where $x = 1$.

Now $I_{n+\frac{1}{2}}(1) = 0$. This condition, therefore, gives us no information as to how the functions I enter the expression. But it shows that if

$$\chi = \Sigma B_n H_{n+\frac{1}{2}}(x) \cos(n\xi + \alpha_n),$$

then

$$2(1 - \cos\xi)\frac{d\chi}{d\xi} - \chi \sin\xi = 0,$$

when

$$x = 1,$$

for all values of ξ.

In the first place, omit α_n.

We find

$$\begin{aligned} &2(1 - \cos\xi) \,.\, 0 - \sin\xi B_0 H_{\frac{1}{2}} \\ &- 2(1 - \cos\xi)\, B_1 H_{\frac{3}{2}} \sin\xi - \sin\xi B_1 H_{\frac{3}{2}} \cos\xi \\ &+ \ldots \\ &- 2(1 - \cos\xi)\, B_n H_{n+\frac{1}{2}}\, n \sin n\xi - \sin\xi B_n H_{n+\frac{1}{2}} \cos n\xi \\ &+ \text{\&c., } ad.\ inf. \quad = 0, \end{aligned}$$

the argument 1 being understood in the functions H.

Hence equating to zero the coefficient of $\sin n\xi$,

$$(2n - 3)\, B_{n-1} H_{n-\frac{1}{2}} - 4n B_n H_{n+\frac{1}{2}} + (2n + 3)\, B_{n+1} H_{n+\frac{3}{2}} = 0,$$

with one exception, viz.,

$$- 2B_0 H_{\frac{1}{2}} - 4B_1 H_{\frac{3}{2}} + 5B_2 H_{\frac{5}{2}} = 0.$$

But

$$H_{n+\frac{1}{2}}(1) = -\frac{1}{n^2 - \frac{1}{4}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (45\text{B}),$$

whence

$$\frac{-B_n + B_{n+1}}{2n+1} = \frac{B_n - B_{n-1}}{2n-1} = \ldots = \frac{B_1 - 2B_0}{3}.$$

Whence $B_n = 2B_0 + \beta n^2$, where B_0, β are arbitrary.

Similarly, if we supposed χ to consist of terms $B'_n H_{n+\frac{1}{2}}(x) \sin n\xi$, we should find $B'_0 = 0$, $B'_1 = 0$, and, therefore, the whole series B'_n, $= 0$.

Hence, if the functions H occur in ψ, they must be in the form of the series

$$\frac{1}{(x-\cos\xi)^{\frac{1}{2}}}\Big\{\alpha\left[\tfrac{1}{2}H_{\frac{1}{2}}(x) + H_{\frac{3}{2}}(x)\cos\xi + H_{\frac{5}{2}}(x)\cos 2\xi + \ldots\right]$$
$$+ \beta\left[H_{\frac{3}{2}}(x)\cos\xi + 2^2 H_{\frac{5}{2}}(x)\cos 2\xi + \ldots\right]\Big\},$$

that is, by p. 484,

$$\alpha + \frac{\pi}{4}\beta(\varpi^2 - 1).$$

The constant term, $\alpha - \pi\beta/4$, is irrelevant; the term $\pi\beta/4\,\varpi^2$ implies a finite velocity at infinity, viz., $\pi\beta/2$, from left to right.

Chapter VI.—Rotational Motion.

Among the problems on rotational motion in liquids, those are probably of most interest in which the circumstances most closely resemble nature, those, namely, in which the spin is due to the internal friction of the liquid. Certain cases of rotational motion in perfect fluids have been discussed by Mr. Hicks ('Phil. Trans.,' 1885), in connection with the theory of the motion of vortex rings, in terms of the functions I have before alluded to (p. 473). I confine myself, in the present chapter, to the consideration of certain cases of motion in viscous liquid.

When the motion in a viscous liquid is slow, the equations may be reduced to the forms, (Basset, 'Hydrodynamics,' § 473)

$$\frac{d\xi}{dt} = \nu\nabla^2\xi; \qquad \frac{d\eta}{dt} = \nu\nabla^2\eta; \qquad \frac{d\zeta}{dt} = \nu\nabla^2\zeta;$$

where ξ, η, ζ are the components of spin parallel to fixed axes, and $\nu = \mu/\rho$, the kinematic coefficient of viscosity. If the motion is symmetrical about the axis of z, we may put $\zeta = 0$, $\xi = -\omega\sin\phi$, $\eta = \omega\cos\phi$, where $\omega = -1/2\varpi\,D\psi$ is the resultant spin, and ϕ is an azimuthal angle measured about the axis of z. But we have seen, in the introduction (p. 452), that

$$\frac{\sin\phi}{\varpi}DV = \nabla^2\frac{\sin\phi}{\varpi}V \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4);$$

write $2\varpi\omega$ for V, and we get

$$D^2\psi = \frac{1}{\nu}\frac{d}{dt}D\psi,$$

Stokes's well-known equation. ('Cambridge Phil. Trans.,' vol. 9, Pt. 2.)

Hence when the motion is steady, ψ is a solution of

$$D^2\psi = 0 \quad \ldots\ldots\ldots\ldots \quad (3d).$$

Or we may assume

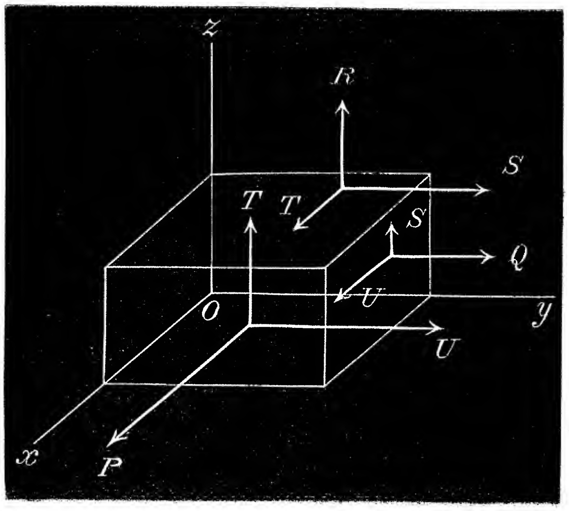
$$\left.\begin{array}{l} D\psi = V \\ \text{where}\quad DV = 0 \end{array}\right\}, \quad \ldots\ldots\ldots\ldots \quad (3e)$$

so that ψ consists of a function of the form of V, together with a particular integral of (3e).

When the boundary is a symmetrical obstacle, the conditions which must be fulfilled there are :—

(1) The velocity normal to the obstacle shall be zero.

(2) The tangential velocity shall be proportional to the stress in the same direction, per unit area, across the surface of the obstacle.



Now if u, v, w are the velocities at any point parallel to fixed rectangular axes, whose origin is at the point, and if P, Q, R, S, T, U denote the stresses per unit area in the fluid, at the point, in the usual manner, as illustrated by the accompanying figure, then, it is known (see Basset, 'Hydrodynamics,' vol. 2, p. 242)

$$P = -p - \tfrac{2}{3}\mu\theta + 2\mu\frac{du}{dx} = -p - \tfrac{2}{3}\mu\theta + 2\mu e$$

$$Q = -p - \tfrac{2}{3}\mu\theta + 2\mu\frac{dv}{dy} = -p - \tfrac{2}{3}\mu\theta + 2\mu f$$

$$\mathrm{R} = -p - \tfrac{2}{3}\mu\theta + 2\mu\frac{dw}{dz} = -p - \tfrac{2}{3}\mu\theta + 2\mu g$$

$$\mathrm{S} = \mu\left(\frac{dw}{dy} + \frac{dv}{dz}\right) = 2\mu a$$

$$\mathrm{T} = \mu\left(\frac{du}{dz} + \frac{dw}{dx}\right) = 2\mu b$$

$$\mathrm{U} = \mu\left(\frac{dv}{dx} + \frac{du}{dy}\right) = 2\mu c,$$

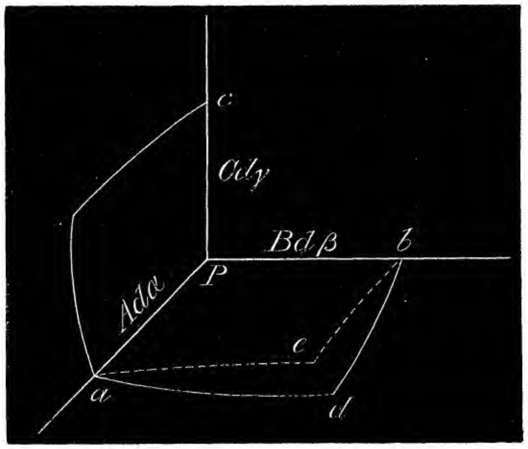
where

$$\theta = \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (62).$$

I will now write these in terms of α, β, γ, the parameters of three orthogonal surfaces, whose intersection may be a convenient determination of the point.

Let

$$dx^2 + dy^2 + dz^2 = \mathrm{A}^2 d\alpha^2 + \mathrm{B}^2 d\beta^2 + \mathrm{C}^2 d\gamma^2.$$



Then, if θ_1, θ_2, θ_3 are the angles through which the normals at α, β, γ to the three surfaces must be rotated about themselves to make them parallel to the normals at $\alpha + d\alpha$, $\beta + d\beta$, $\gamma + d\gamma$, it is not hard to see

$$\theta_3 = \frac{\mathrm{B}}{\mathrm{A}d\alpha}\,d\beta - \frac{\mathrm{A}}{\mathrm{B}d\beta}\,d\alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (63),$$

with corresponding expressions for θ_1, θ_2. For, in passing from P to b, along the normal to the surface β, the normal to the surface α is rotated through the small angle *ebd*, or $\frac{d\mathrm{B}}{\mathrm{A}d\alpha}\,d\beta$, and, in passing from b to d, it is rotated through the angle *ead* or $\frac{d\mathrm{A}}{\mathrm{B}d\beta}\,d\alpha$, in the negative sense.

But the change in u, the velocity parallel to the normal at α, β, γ to the surface α, as we pass from α, β, γ to $\alpha + d\alpha,\ \beta + d\beta,\ \gamma + d\gamma$, is

$$\frac{du}{d\alpha} d\alpha + \frac{du}{d\beta} d\beta + \frac{du}{d\gamma} d\gamma - v\theta_3 + w\theta_2$$

$$= d\alpha \left[\frac{du}{d\alpha} + \frac{v}{\mathrm{B}} \frac{d\mathrm{A}}{d\beta} + \frac{w}{\mathrm{C}} \frac{d\mathrm{A}}{d\gamma}\right] + d\beta \left[\frac{du}{d\beta} - \frac{v}{\mathrm{A}} \frac{d\mathrm{B}}{d\alpha}\right] + d\gamma \left[\frac{du}{d\gamma} - \frac{w}{\mathrm{A}} \frac{d\mathrm{C}}{d\alpha}\right].$$

Hence we get

$$e = \frac{1}{\mathrm{A}} \left[\frac{du}{d\alpha} + \frac{v}{\mathrm{B}} \frac{d\mathrm{A}}{d\beta} + \frac{w}{\mathrm{C}} \frac{d\mathrm{A}}{d\gamma}\right]$$

$$2a = \frac{1}{\mathrm{C}} \frac{dv}{d\gamma} + \frac{1}{\mathrm{B}} \frac{dw}{d\beta} - \frac{w}{\mathrm{BC}} \frac{d\mathrm{C}}{d\beta} - \frac{v}{\mathrm{BC}} \frac{d\mathrm{B}}{d\gamma} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (64),$$

with four others, which may be written down from symmetry.

These enable us to express the second boundary condition (p. 494) in terms of any coordinates which we may wish to employ.

Let us consider the solution of ($3e$) in terms of the coordinates r, θ.

We may assume, p. 489,

$$\mathrm{V} = \sum_{n=0}^{n=\infty} (\mathrm{A}_n r^n + \mathrm{B}_n r^{-n+1})\, \mathrm{I}_n (\cos\theta),$$

and

$$\mathrm{D} = \frac{d^2}{dr^2} + \frac{1-\mu^2}{r^2} \frac{d^2}{d\mu^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6),$$

where $\mu = \cos\theta$.

Hence we require a particular integral of the equation,

$$\frac{d^2\psi_n}{dr^2} - \frac{n(n-1)}{r^2} \psi_n = \mathrm{A}_n r^n + \mathrm{B}_n r^{-n+1}.$$

Clearly

$$\psi_n = \frac{\mathrm{A}_n r^{n+2} - \mathrm{B}_n r^{-n+3}}{2(2n-1)}$$

is a solution.

Hence, a particular integral of ($3e$) is, changing the arbitrary constants,

$$\psi = \sum_{n=0}^{n=\infty} (\mathrm{A}_n r^{n+2} + \mathrm{B}_n r^{-n+3})\, \mathrm{I}_n (\cos\theta),$$

and, omitting terms, which, by the expressions (p. 453),

$$\mathrm{R} = \frac{1}{r^2 \sin\theta} \frac{d\psi}{d\theta}$$

$$\Theta = -\frac{1}{r\sin\theta} \frac{d\psi}{dr}$$

may be seen to give infinite velocities, we may write as a value of ψ, capable of expressing any slow, steady motion in a region of viscous liquid from which the origin is excluded,

$$\psi = \mathrm{D}_1 \cos\theta + (\mathrm{C}_2 r^2 + \mathrm{D}_2 r^{-1} + \mathrm{B}_2 r)\,\mathrm{I}_2(\cos\theta) + \sum_{n=3}^{n=\infty} (\mathrm{D}_n r^{-n+1} + \mathrm{B}_n r^{-n+3})\,\mathrm{I}_n(\cos\theta) \quad . \quad . \quad . \quad . \quad (65).$$

An expression equivalent to this is given by Mr. BUTCHER ('London Math. Soc. Proc.,' vol. 8).

In terms of the coordinates r, θ, I shall consider two cases of motion, viz., (1) when the obstacle is a sphere, (2) when it departs little from sphericity. When there is no slipping, the boundary conditions are $\mathrm{R} = 0$, $\Theta = 0$; when there is slipping the conditions that hold at the boundary are not easily expressed in the second case; in its discussion I shall, therefore, confine myself to the supposition that the velocity at the boundary is zero.

1. *The Obstacle a Sphere.*

We find from (64)

$$2c = \frac{d\mathrm{R}}{rd\theta} + \frac{d\Theta}{dr} - \frac{\Theta}{r},$$

and, consequently, if a be the radius, we must have, when $r = a$,

$$\mathrm{R} = 0,$$

$$\mu\left(\frac{d\mathrm{R}}{r\,d\theta} + \frac{d\Theta}{dr} - \frac{\Theta}{r}\right) = \beta\Theta,$$

or, written in terms of ψ,

$$\frac{d\psi}{d\theta} = 0$$

$$-(1-\mu^2)\frac{d^2\psi}{d\mu^2} + r^2\frac{d^2\psi}{dr^2} - 2r\frac{d\psi}{dr} = \frac{\beta}{\mu} r^2 \frac{d\psi}{dr},$$

where β is a constant, which is infinite if there is no slipping, and $\mu = \cos\theta$.

But

$$-(1-\mu^2)\frac{d^2}{d\mu^2}\,\mathrm{I}_n(\mu) = n(n-1)\,\mathrm{I}_n(\mu).$$

Hence the above conditions require $\mathrm{D}_n = \mathrm{B}_n = 0$, except when $n = 2$, in which case we have

$$\mathrm{C}_2 a^2 + \mathrm{D}_2 a^{-1} + \mathrm{B}_2 a = 0,$$

$$2\mathrm{C}_2 a^2 - \mathrm{D}_2 a^{-1}\left(1 + \frac{6\mu}{a\beta}\right) + \mathrm{B}_2 a = 0.$$

These give

$$\psi = \mathrm{C}_2 \left[r^2 - a^2 \frac{3\left(1 + \dfrac{2\mu}{\beta a}\right)\dfrac{r}{a} - \dfrac{a}{r}}{2\left(1 + \dfrac{3\mu}{\beta a}\right)} \right] \mathrm{I}_2 (\cos\theta) \quad . \quad . \quad . \quad . \quad . \quad (66).$$

It is clear that $\mathrm{C}_2 = \mathrm{V}$, the velocity from right to left at infinity.

Passing from this well-known result, which is here given for reference and illustration, let us consider the motion about an approximately spherical, symmetrical obstacle, at whose surface no slipping takes place.

Let $r = a[1 + f(\theta)]$ be the equation of the surface, and let $f(\theta)$ be developed by Chapter II. into the form $\Sigma \alpha_n \mathrm{I}_n (\cos\theta)$.

Let us consider the surface

$$r = a[1 + \alpha_n \mathrm{I}_n (\cos\theta)],$$

where we shall suppose α_n so small that its square may be neglected.

We must have $\dfrac{d\psi}{dr} = 0$, $\dfrac{d\psi}{d\theta} = 0$ in (65) when $r = a[1 + \alpha_n \mathrm{I}_n (\cos\theta)]$; and since the motion will not be far different from that in the case when the surface is a perfect sphere, all the cofficients which occur are of the order of α_n, excepting C_2, D_2, B_2, and, therefore, except where these coefficients enter, we may disregard the departure from a spherical form. We find, omitting the argument $\cos\theta$, and writing D'_n, B'_n, for $\mathrm{D}_n a^{-n+1}$, $\mathrm{B}_n a^{-n+3}$ respectively,

$$\begin{aligned} a\frac{d\psi}{dr} &= [(2\mathrm{C}_2 a^2 - \mathrm{D}_2 a^{-1} + \mathrm{B}_2 a) + \alpha_n \mathrm{I}_n (2\mathrm{C}_2 a^2 + 2\mathrm{D}_2 a)]\, \mathrm{I}_2 \\ &\qquad + \Sigma[-(r-1)\mathrm{D}'_r - (r-3)\mathrm{B}'_r]\, \mathrm{I}_r \\ &= 0. \end{aligned}$$

$$\begin{aligned} \frac{d\psi}{d(\cos\theta)} &= [(\mathrm{C}_2 a^2 + \mathrm{D}_2 a^{-1} + \mathrm{B}_2 a) + \alpha_n \mathrm{I}_n (2\mathrm{C}_2 a^2 - \mathrm{D}_2 a^{-1} + \mathrm{B}_2 a)] \cos\theta \\ &\qquad + \Sigma[\mathrm{D}'_r + \mathrm{B}'_r]\, \mathrm{P}_{r-1} \\ &= 0. \end{aligned}$$

Hence

$$\mathrm{C}_2 a^2 + \mathrm{D}_2 a^{-1} + \mathrm{B}_2 a = 0,$$
$$2\mathrm{C}_2 a^2 - \mathrm{D}_2 a^{-1} + \mathrm{B}_2 a = 0.$$

Making use of these results we find

$$\Sigma[\mathrm{D}'_r + \mathrm{B}'_r]\, \mathrm{P}_{r-1} = 0\,;$$

whence

$$\mathrm{D}'_r + \mathrm{B}'_r = 0$$

for all values of r, and

$$-\alpha_n \mathrm{I}_n\, 2\mathrm{B}_2 a\, \mathrm{I}_2 + \Sigma\, 2\mathrm{B}'_r\, \mathrm{I}_r = 0.$$

But

$$\mathrm{I}_n\,\mathrm{I}_2 = \frac{\delta_n}{2}\,\mathrm{I}_{n+2} + \frac{\epsilon_n - 1}{2}\,\mathrm{I}_n + \frac{\zeta_n}{2}\,\mathrm{I}_{n-2}$$

where δ_n, ϵ_n, ζ_n are given by (17), p. 460.

Therefore

$$\mathrm{B}'_{n+2} = \mathrm{B}_2 a\,\alpha_n \frac{\delta_n}{2},$$

$$\mathrm{B}'_{n} = \mathrm{B}_2 a\,\alpha_n \frac{\epsilon_n - 1}{2},$$

$$\mathrm{B}'_{n-2} = \mathrm{B}_2 a\,\alpha_n \frac{\zeta_n}{2},$$

all the others vanishing.

Hence

$$\begin{aligned}\psi = &- \frac{\mathrm{V}a^2}{2}\,\mathrm{I}_2(\cos\theta)\left[\frac{3r}{a} - \frac{a}{r} - \frac{2r^2}{a^2}\right] \\ &- \frac{3\mathrm{V}a^2}{2}\,\alpha_n\,\frac{\zeta_n}{2}\,\mathrm{I}_{n-2}(\cos\theta)\left[\frac{a^{n-5}}{r^{n-5}} - \frac{a^{n-3}}{r^{n-3}}\right] \\ &- \frac{3\mathrm{V}a^2}{2}\,\alpha_n\,\frac{\epsilon_n - 1}{2}\,\mathrm{I}_n(\cos\theta)\left[\frac{a^{n-3}}{r^{n-3}} - \frac{a^{n-1}}{r^{n-1}}\right] \\ &- \frac{3\mathrm{V}a^2}{2}\,\alpha_n\,\frac{\delta_n}{2}\,\mathrm{I}_{n+2}(\cos\theta)\left[\frac{a^{n-1}}{r^{n-1}} - \frac{a^{n+1}}{r^{n+1}}\right] \quad . \quad . \quad . \quad . \quad . \quad (67)\end{aligned}$$

From this formula the solution for the case $r = a\,\{1 + \Sigma\alpha_n\,\mathrm{I}_n(\cos\theta)\}$ can be built up.

2. *Spheroids.*

I will now consider the solution of (3*e*), p. 494, in terms of co-ordinates, p, q, defined on p. 477.

By p. 479, we may write

$$\mathrm{V} = \sum_{n=o}^{n=\infty}\left[\mathrm{A}_n\mathrm{I}_n(p) + \mathrm{B}_n\mathrm{H}_n(p)\right]\mathrm{I}_n(q),$$

and

$$\mathrm{D} = \frac{-1}{h^2(p^2 - q^2)}\left[(1 - p^2)\frac{d^2}{dp^2} - (1 - q^2)\frac{d^2}{dq^2}\right] \quad . \quad . \quad . \quad . \quad . \quad (6a).$$

Hence

$$(1 - p^2)\frac{d^2\psi}{dp^2} - (1 - q^2)\frac{d^2\psi}{dq^2} = -h^2\sum_{n=o}^{n=\infty}(p^2 - q^2)\left[\mathrm{A}_n\mathrm{I}_n(p) + \mathrm{B}_n\mathrm{H}_n(p)\right]\mathrm{I}_n(q) \quad . \quad (68).$$

But, when n is not less than 4,

$$p^2\,\frac{\mathrm{I}_n(p)}{\mathrm{H}_n(p)} = \delta_n\,\frac{\mathrm{I}_{n+2}(p)}{\mathrm{H}_{n+2}(p)} + \epsilon_n\,\frac{\mathrm{I}_n(p)}{\mathrm{H}_n(p)} + \zeta_n\,\frac{\mathrm{I}_{n-2}(p)}{\mathrm{H}_{n-2}(p)}$$

by pp. 460, 469.

Hence

$$(p^2 - q^2)\, A_n I_n(p)\, I_n(q)$$
$$= A_n \delta_n \left[I_{n+2}(p)\, I_n(q) - I_n(p)\, I_{n+2}(q)\right] + A_n \zeta_n \left[I_{n-2}(p)\, I_n(q) - I_n(p)\, I_{n-2}(q)\right].$$

The corresponding terms in a particular integral of (68) are, by p. 480,

$$\frac{h^2 A_n \delta_n}{2(2n+1)} \left[I_{n+2}(p)\, I_n(q) + I_n(p)\, I_{n+2}(q)\right]$$
$$\frac{h^2 A_n \zeta_n}{2(2n-3)} \left[I_n(p)\, I_{n-2}(q) + I_{n-2}(p)\, I_n(q)\right].$$

Hence the terms arising in a particular integral of (68) from the terms on the right for which n is not less than 4, may be written

$$\Sigma \left\{\mathfrak{A}_{n+2}\left[I_{n+2}(p)\, I_n(q) + I_{n+2}(q)\, I_n(p)\right] + \mathfrak{B}_{n+2}\left[H_{n+2}(p)\, I_n(q) + I_{n+2}(q)\, H_n(p)\right]\right\},$$

where

$$\mathfrak{A}_{n+2} = \frac{h^2}{2(2n+1)} \left[A_n \delta_n - A_{n+2}\, \zeta_{n+2}\right]$$
$$\mathfrak{B}_{n+2} = \frac{h^2}{2(2n+1)} \left[B_n \delta_n - B_{n+2}\, \zeta_{n+2}\right].$$

The terms on the right of (68), when n is less than 4, are

$$(p^2 - q^2) \cdot \left\{A_0 + B_0 p + [A_1 p + B_1]\, q + [A_2 I_2(p) + B_2 H_2(p)]\, I_2(q) \right.$$
$$\left. + [A_3 I_3(p) + B_3 H_3(p)]\, I_3(q)\right\}$$
$$= 2A_0\, I_2(p) + 2B_0 I_3(p) - 2[A_0 + B_0 p] I_2(q) + 2[A_1 I_3(p) + B_1 I_2(p)] q - 2[A_1 p + B_1] I_3(q)$$
$$+ A_2 \delta_2 \left[I_4(p)\, I_2(q) - I_2(p)\, I_4(q)\right] + B_2\left\{\delta_2 \left[H_4(p) I_2(q) - H_2(p)\, I_4(q)\right] - \frac{p}{3} I_2(q)\right\}$$
$$+ A_3 \delta_3 \left[I_5(p)\, I_3(q) - I_3(p)\, I_5(q)\right] + B_3 \left\{\delta_3 \left[H_5(p)\, I_3(q) - H_3(p)\, I_5(q)\right] - \tfrac{1}{15} \cdot I_3(q)\right\}.$$

Thus, the series of terms we have already found to occur in the particular integral, of (68), continues regularly down to $n = 2$, while there are also found the irregular terms

$$- A_0 I_2(p) - \frac{B_0}{3} I_3(p)$$
$$+ \left[- B_1 I_2(p) - \frac{A_1}{3} I_3(p)\right] q$$
$$+ \left[A_0 + \left(B_0 + \frac{B_2}{6}\right) p\right] I_2(q)$$
$$+ \left[\left(\frac{B_1}{3} + \frac{B_3}{45}\right) + \frac{A_1}{3} p\right] I_3(q).$$

Thus ψ consists of these terms, together with the series

$$\sum_{n=2}^{n=\infty}\{\mathfrak{A}_{n+2}[\mathrm{I}_{n+2}(p)\,\mathrm{I}_n(q)+\mathrm{I}_n(p)\,\mathrm{I}_{n+2}(q)]+\mathfrak{B}_{n+2}[\mathrm{H}_{n+2}(p)\,\mathrm{I}_n(q)+\mathrm{H}_n(p)\,\mathrm{I}_{n+2}(q)]\},$$

and the series

$$\sum_{n=0}^{n=\infty}[\mathrm{C}_n\mathrm{I}_n(p)+\mathrm{D}_n\mathrm{H}_n(p)]\,\mathrm{I}_n(q).$$

All the constants above are independent. But referring to p. 490, we see that the conditions for finite and continuous velocity everywhere require that A_0, B_0, A_1, B_1; $\mathfrak{A}_4$, $\mathfrak{A}_5$, &c.; C_0, D_0, C_3, C_4, &c., should all vanish. Hence all legitimate expressions of ψ are included in

$$\begin{aligned}\psi = \mathrm{C}_0 &+ \mathrm{D}_1 q\\ &+\mathrm{I}_2(q)\,[\mathrm{B}_2 p+\mathrm{C}_2\mathrm{I}_2(p)+\mathrm{D}_2\mathrm{H}_2(p)+\mathfrak{B}_4\mathrm{H}_4(p)]\\ &+\mathrm{I}_3(q)\,[\mathrm{B}_3+\mathrm{D}_3\mathrm{H}_3(p)+\mathfrak{B}_5\mathrm{H}_5(p)]\\ &+\mathrm{I}_4(q)\,[\mathfrak{B}_4\mathrm{H}_2(p)+\mathrm{D}_4\mathrm{H}_4(p)+\mathfrak{B}_6\mathrm{H}_6(p)]\\ &+\text{\&c., regularly} \quad \ldots\ldots\ldots \quad (69),\end{aligned}$$

writing B_2, B_3 in place of $\frac{\mathrm{B}_2}{6}$, $\frac{\mathrm{B}_3}{45}$ respectively.

The first case to which I shall apply this formula is that in which an obstacle in the shape of a spheroid is opposed to the flow of liquid, and, to begin with, I shall suppose there is no slipping at the surface; so that the boundary conditions are

$$\frac{d\psi}{dp}=0 \quad \frac{d\psi}{dq}=0,$$

when $p=p_0$, where p_0 is the parameter of the fixed surface.

These require, omitting the argument, p_0,

$$\begin{gathered}\mathrm{D}_1=0\\ \mathrm{B}_2p_0+\mathrm{C}_2\mathrm{I}_2+\mathrm{D}_2\mathrm{H}_2+\mathfrak{B}_4\mathrm{H}_4=0=\mathrm{B}_2+\mathrm{C}_2\mathrm{P}_1+\mathrm{D}_2\mathrm{Q}_1+\mathfrak{B}_4\mathrm{Q}_3\\ \mathrm{B}_3+\mathrm{D}_3\mathrm{H}_3+\mathfrak{B}_5\mathrm{H}_5=0=\mathrm{D}_3\mathrm{Q}_2+\mathfrak{B}_5\mathrm{Q}_4\\ \mathfrak{B}_4\mathrm{H}_2+\mathrm{D}_4\mathrm{H}_4+\mathfrak{B}_6\mathrm{H}_6=0=\mathfrak{B}_4\mathrm{Q}_1+\mathrm{D}_4\mathrm{Q}_3+\mathfrak{B}_6\mathrm{Q}_5\\ \ldots\ldots\ldots\end{gathered}$$

Hence, three of the above constants are arbitrary, namely, we may give what values we please to C_2, $\mathfrak{B}_4$, B_3, and all the others will then be determinate. If we give a finite value to $\mathfrak{B}_4$ or B_3 we get an infinite series which in each case is divergent (see p. 508). Let us then first consider the solution given by $\mathfrak{B}_4=0$, $\mathrm{B}_3=0$.

We have

$$\psi = \mathrm{I}_2(q)\,[\mathrm{C}_2\mathrm{I}_2(p) + \mathrm{D}_2\mathrm{H}_2(p) + \mathrm{B}_2 p],$$

where

$$\mathrm{C}_2 p_0 + \mathrm{D}_2\,\mathrm{Q}_1(p_0) + \mathrm{B}_2 = 0$$

$$\mathrm{C}_2\mathrm{I}_2(p_0) + \mathrm{D}_2\mathrm{H}_2(p_0) + \mathrm{B}_2 p_0 = 0.$$

Hence, since

$$\mathrm{H}_2(p_0) = \frac{p_0^2 - 1}{4}\log\frac{p_0 + 1}{p_0 - 1} - \frac{p_0}{2},$$

$$\mathrm{Q}_1(p_0) = \frac{p_0}{2}\log\frac{p_0 + 1}{p_0 - 1} - 1,$$

we get

$$\mathrm{D}_2 = -\frac{p_0^2 + 1}{2\mathrm{L}(p_0)}\,\mathrm{C}_2,$$

$$\mathrm{B}_2 = -\frac{1}{2\mathrm{L}(p_0)}\,\mathrm{C}_2,$$

where

$$\mathrm{L}(p_0) = \frac{p_0^2 + 1}{4}\log\frac{p_0 + 1}{p_0 - 1} - \frac{p_0}{2}.$$

Now the term which gives a finite velocity at infinity is $\mathrm{C}_2\mathrm{I}_2(p)\,\mathrm{I}_2(q) = -\,\mathrm{C}_2\varpi^2/4h^2$. Hence, if this velocity be V from right to left, $\mathrm{C}_2 = 2h^2\mathrm{V}$, and

$$\psi = 2h^2\mathrm{V}\;\mathrm{I}_2(q)\left[\mathrm{I}_2(p) - \frac{p + (p_0^2 + 1)\,\mathrm{H}_2(p)}{2\mathrm{L}(p_0)}\right] \quad . \quad . \quad . \quad . \quad . \quad (70).$$

From this result we find, by simple reductions, that the velocities U, W (p. 478), perpendicular to the axis and parallel to it, are

$$\mathrm{U} = \frac{\mathrm{V}}{2\mathrm{L}(p_0)}\,\frac{q\,(p^2 - p_0^2)\sqrt{(1 - q^2)}}{(p^2 - q^2)\sqrt{(p^2 - 1)}},$$

$$\mathrm{W} = \frac{\mathrm{V}}{2\mathrm{L}(p_0)}\left\{\frac{1 + p_0^2}{2}\log\frac{p + 1}{p - 1} + \frac{p\,(q^2 - p_0^2)}{p^2 - q^2}\,2\mathrm{L}(p_0)\right\} \quad . \quad . \quad . \quad (71).$$

From (70), by the equation $\psi =$ const., we get the forms of the stream surfaces, and from this point of view, an integral which includes (70) has been given by MR. HERMAN, 'Quarterly Journal of Mathematics,' 1889, No. 92. Taking OBERBECK's values for the velocities due to the steady motion of an ellipsoid through a viscous liquid ('Crelle,' vol. 81), he finds as one surface on which the stream-lines relative to the ellipsoid lie,

$$\left[\int^{\epsilon}\left(1 + \frac{c^2}{c^2 + \lambda}\right)\frac{d\lambda}{\{(a^2 + \lambda)(b^2 + \lambda)(c^2 + \lambda)\}^{\frac{1}{2}}} - (\mathrm{A}_\epsilon + \mathrm{B}_\epsilon + \mathrm{C}_\epsilon)\,\epsilon\right]xy = \text{constant},$$

where

$$\mathrm{A}_\epsilon = \int_\epsilon^\infty \frac{d\lambda}{(a^2+\lambda)^{\frac{3}{2}}(b^2+\lambda)^{\frac{1}{2}}c^2+\lambda)^{\frac{1}{2}}},$$

and $(a^2+\epsilon)^{\frac{1}{2}}$ is the semi-major axis of the confocal to the moving ellipsoid, which passes through the point x, y, z. I have slightly changed his notation, making the axis of z that along which the ellipsoid is moving.

But if the ellipsoid is one of revolution, so that $a = b$, then

$$h^3\mathrm{A}_\epsilon = h^3\mathrm{B}_\epsilon = \frac{p}{p^2-1} - \tfrac{1}{2}\log\frac{p+1}{p-1}$$

$$h^3\mathrm{C}_\epsilon = \log\frac{p+1}{p-1} - \frac{2}{p}$$

$$h\int_\epsilon^\infty \frac{d\lambda}{\{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)\}^{\frac{1}{2}}} = \log\frac{p+1}{p-1} = h\mathrm{H}_\epsilon$$

in the notation of p. 477, where

$$c^2 + \epsilon = h^2p^2.$$

Hence, writing Mr. HERMAN'S integral

$$[\mathrm{H}_0 + c^2\mathrm{C}_0 - \mathrm{H}_\epsilon - c^2\mathrm{C}_\epsilon - \epsilon(\mathrm{A}_\epsilon + \mathrm{B}_\epsilon + \mathrm{C}_\epsilon)]\,xy = \text{const.},$$

and substituting, we find the expression in brackets equal to

$$\frac{1}{h}\left\{4\mathrm{L}(p_0) - 2(p_0^2+1)\left[\tfrac{1}{2}\log\frac{p+1}{p-1} - \frac{p}{{}^2-1}\right] - \frac{4p}{p^2-1}\right\},$$

and

$$xy = \varpi^2 \sin\phi\cos\phi,$$
$$= h^2(p^2-1)(1-q^2)\sin\phi\cos\phi,$$

ϕ being the azimuthal angle about the axis.

That is to say, the surfaces

$$[2\mathrm{L}(p_0)\,\mathrm{I}_2(p) - (p_0^2+1)\,\mathrm{H}_2(p) - p]\,\mathrm{I}_2(q)\sin\phi\cos\phi = \text{const.},$$

are composed of stream-lines; and the motion being symmetrical about the axis, $\phi = $ const. is the other family of surfaces composed of stream-lines. This gives the expression (70) equated to a constant, as representing stream surfaces.

From the expression (70), we can obtain the expression (66), when β is infinite, giving ψ when the obstacle is a sphere. We must suppose h indefinitely small; then we find $p = \frac{r}{h}$, $q = \cos\theta$, where θ is the angle between the asymptotes of the hyperbola q, and is ultimately the vectorial angle of any point. But by (28), when h is small,

$$\mathrm{H}_2(p) = -\tfrac{1}{3}\frac{h}{r},$$

$$\mathrm{L}(p_0) = \mathrm{H}_2(p_0) + \tfrac{1}{2}\log\frac{p_0+1}{p_0-1} = \tfrac{2}{3}\frac{h}{a}$$

$$p + (p_0^2+1)\,\mathrm{H}_2(p) = \frac{1}{h}\left[r - \frac{a^2}{3r}\right]$$

and

$$\psi = h^2\mathrm{V}\sin^2\theta\left[\frac{r - \dfrac{a^2}{3r}}{h}\,\frac{3a}{2h} - \frac{r^2}{2h^2}\right]$$

$$= \frac{a^2\mathrm{V}\sin^2\theta}{4}\left[\frac{3r}{a} - \frac{a}{r} - \frac{2r^2}{a^2}\right]$$

which is the expression of p. 498, in which β is supposed infinite.

Three particular cases, besides the sphere, are worthy of notice.

(1.) *The obstacle a paraboloid,* of parameter $2l_0$. Taking $2l$, $2m$ as the parameters of the confocal paraboloids through any point, the solution is found by making h very great in (70).

We find, without difficulty,

$$\psi = \frac{\mathrm{V}}{2\log 4h/l_0}\,m\left[l\log\frac{l_0}{l} + l - l_0\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (72).$$

(2) *A planetary spheroid of eccentricity* $1/\sqrt{2}$. Hence, by p. 478,

$$p_0 = i,$$

and (70) becomes

$$\psi = a^2\mathrm{V}\,\mathrm{I}_2(q)\left[\mathrm{I}_2(p) + \frac{p}{i}\right],$$

$$= \frac{a^2\mathrm{V}}{4}(q^2-1)(p-i)^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (73)$$

where $2a$ is the greater axis.

(3) *A circular disc.* Here $p_0 = \mathbf{0}$,

$$\psi = -2a^2\mathrm{V}\,\mathrm{I}_2(q)\left[\mathrm{I}_2(p) + \frac{2i}{\pi}\{p + \mathrm{H}_2(p)\}\right] \quad . \quad . \quad . \quad . \quad (74).$$

where a is the radius.

Turning to (51) we find the velocities given by this formula are

$$\mathrm{U} = \frac{2\mathrm{V}}{i\pi}\,\frac{qp^2\sqrt{(1-q^2)}}{(p^2-q^2)\sqrt{(p^2-1)}},$$

$$\mathrm{W} = \frac{2\mathrm{V}}{i\pi}\left[\tfrac{1}{2}\log\frac{p+1}{p-1} + \frac{pq^2}{p^2-q^2} - \frac{i\pi}{2}\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (75)$$

vanishing at the edge.

As an example of the analysis to which the use of the expression leads, I will now find the value of ψ for a surface approximately spheroidal, say $p = p_0 + \alpha_n \mathrm{I}_n(q)$, where α_n is a small quantity whose square will be neglected.

First suppose n an even number; let us then assume

$$\begin{aligned}\psi = {} & \mathrm{I}_2(q)\,[\mathrm{B}_2 p + \mathrm{C}_2\mathrm{I}_2(p) + \mathrm{D}_2\mathrm{H}_2(p) + \mathfrak{B}_4\mathrm{H}_4(p)] \\ & + \mathrm{I}_4(q)\,[\mathfrak{B}_4\mathrm{H}_2(p) + \mathrm{D}_4\mathrm{H}_4(p) + \mathfrak{B}_6\mathrm{H}_6(p)] \\ & + \ldots \\ & + \mathrm{I}_n(q)\,[\mathfrak{B}_n\mathrm{H}_{n-2}(p) + \mathrm{D}_n\mathrm{H}_n(p) + \mathfrak{B}_{n+2}\mathrm{H}_{n+2}(p)] \\ & + \mathrm{I}_{n+2}(q)\,[\mathfrak{B}_{n+2}\mathrm{H}_n(p) + \mathrm{D}_{n+2}\mathrm{H}_{n+2}(p)],\end{aligned}$$

at which point we shall find that we may stop; in fact, there are $n + 3$ constants in the above expression, and $n/2 + 1$ terms, each of which yields two equations corresponding to $d\psi/dp = 0$, $d\psi/dq = 0$; we are thus left with one constant arbitrary, which is determined when we know the velocity at infinity. It is clear that all the constants except B_2, C_2, D_2 are of the order of α_n, and hence, at the surface, we may put $p = p_0$ in the terms where they arise as coefficients.

The condition $\dfrac{d\psi}{dq} = 0$ when $p = p_0 + \alpha_n\mathrm{I}_n(q)$ gives

$$\begin{aligned} & q\,[\mathrm{B}_2 p_0 + \mathrm{C}_2\mathrm{I}_2 + \mathrm{D}_2\mathrm{H}_2 + \mathfrak{B}_4\mathrm{H}_4] \\ & \quad + \alpha_n q\mathrm{I}_n(q)\,[\mathrm{B}_2 + \mathrm{D}_2\mathrm{P}_1 + \mathrm{D}_2\mathrm{Q}_1] \\ & \quad + \mathrm{P}_3(q)\,[\mathfrak{B}_4\mathrm{H}_2 + \mathrm{D}_4\mathrm{H}_4 + \mathfrak{B}_6\mathrm{H}_6] \\ & \quad + \ldots \\ & \quad + \mathrm{P}_{n-1}(q)\,[\mathfrak{B}_n\mathrm{H}_{n-2} + \mathrm{D}_n\mathrm{H}_n + \mathfrak{B}_{n+2}\mathrm{H}_{n+2}] \\ & \quad + \mathrm{P}_{n+1}(q)\,[\mathfrak{B}_{n+2}\mathrm{H}_n + \mathrm{D}_{n+2}\mathrm{H}_{n+2}] = 0;\end{aligned}$$

$\dfrac{d\psi}{dp} = 0$ gives

$$\begin{aligned} & \mathrm{I}_2(q)\,[\mathrm{B}_2 + \mathrm{C}_2\mathrm{P}_1 + \mathrm{D}_2\mathrm{Q}_1 + \mathfrak{B}_4\mathrm{Q}_3] \\ & \quad + \alpha_n\mathrm{I}_2(q)\,\mathrm{I}_n(q)\left[\mathrm{C}_2 + \mathrm{D}_2\frac{d\mathrm{Q}_1(p_0)}{dp_0}\right] \\ & \quad + \mathrm{I}_4(q)\,[\mathfrak{B}_4\mathrm{Q}_1 + \mathrm{D}_4\mathrm{Q}_3 + \mathfrak{B}_6\mathrm{Q}_5] \\ & \quad + \ldots \\ & \quad + \mathrm{I}_n(q)\,[\mathfrak{B}_n\mathrm{Q}_{n-3} + \mathrm{D}_n\mathrm{Q}_{n-1} + \mathfrak{B}_{n+2}\mathrm{Q}_{n+1}] \\ & \quad + \mathrm{I}_{n+2}(q)\,[\mathfrak{B}_{n+2}\mathrm{Q}_{n-1} + \mathrm{D}_{n+2}\mathrm{Q}_{n+1}] = 0.\end{aligned}$$

Now,

$$I_2(q)\, I_n(q) = \frac{\delta_n}{2} I_{n+2}(q) + \frac{\epsilon_n - 1}{2} I_n(q) + \frac{\zeta_n}{2} I_{n-2}(q) \quad . \quad . \quad . \quad . \quad (17),$$

by p. 460.

Hence the second series of equations requires

$$\begin{aligned}
B_2 + C_2P_1 + D_2Q_1 + \mathfrak{B}_4Q_3 &= 0,\\
\mathfrak{B}_4Q_1 + D_4Q_3 + \mathfrak{B}_6Q_5 &= 0,\\
\cdots\cdots\cdots\cdots\cdots\cdots &\\
\mathfrak{B}_{n-2}Q_{n-5} + D_{n-2}Q_{n-3} + \mathfrak{B}_nQ_{n-1} + \alpha\zeta_n &= 0,\\
\mathfrak{B}_nQ_{n-3} + D_nQ_{n-1} + \mathfrak{B}_{n+2}Q_{n+1} + \alpha(\epsilon_n - 1) &= 0,\\
\mathfrak{B}_{n+2}Q_{n-1} + D_nQ_{n+1} \qquad\qquad + \alpha\,\delta_n &= 0,
\end{aligned}$$

where

$$\alpha = \frac{\alpha_n}{2}\left[C_2 + D_2 \frac{dQ_1(p_0)}{dp_0}\right].$$

Making use of the first of these, we see that the former series requires

$$\begin{aligned}
B_2p_0 + C_2I_2 + D_2H_2 + \mathfrak{B}_4H_4 &= 0,\\
\mathfrak{B}_4H_2 + D_4H_4 + \mathfrak{B}_6H_6 &= 0.\\
\cdots\cdots\cdots\cdots\cdots &\\
\mathfrak{B}_nH_{n-2} + D_nH_n + \mathfrak{B}_{n+2}H_{n+2} &= 0,\\
\mathfrak{B}_{n+2}H_n + D_{n+2}H_{n+2} \qquad &= 0,
\end{aligned}$$

Taking any pair of equations

$$\begin{aligned}
\mathfrak{B}_rQ_{r-3} + D_rQ_{r-1} + \mathfrak{B}_{r+2}Q_{r+1} &= 0,\\
\mathfrak{B}_rH_{r-2} + D_rH_r + \mathfrak{B}_{r+2}H_{r+2} &= 0,
\end{aligned}$$

where r lies between 4 and $n - 4$ inclusive, we find

$$D_rV_r = \mathfrak{B}_rU_r = \mathfrak{B}_{r+2}U_{r+2} = \beta, \text{ a constant, say,}$$

where

$$\begin{aligned}
U_r &= H_rQ_{r-3} - H_{r-2}Q_{r-1},\\
V_r &= U_rU_{r+2} / (H_{r-2}Q_{r+1} - H_{r+2}Q_{r-3}).
\end{aligned}$$

In place of these, the last three give :—

Last pair,

$$\mathfrak{B}_{n+2}U_{n+2} = -\,\alpha\delta_nH_{n+2},$$

$$D_{n+2}V_{n+2} = \alpha\delta_n\left[H_{n+2} - \frac{H_{n+4}V_{n+2}}{U_{n+4}}\right]$$

Last but one,

$$\mathfrak{B}_{n+2}U_{n+2} = \mathfrak{B}_n U_n + \alpha(\epsilon_n - 1)H_n,$$

$$D_n V_n = \mathfrak{B}_n U_n - \alpha(\epsilon_n - 1)\frac{H_{n+2}V_n}{U_{n+2}}.$$

Last but two,

$$\begin{aligned}\mathfrak{B}_n U_n &= \mathfrak{B}_{n-2}U_{n-2} + \alpha\zeta_n H_{n-2}, \\ &= \beta + \alpha\zeta_n H_{n-2}, \\ D_{n-2}V_{n-2} &= \beta + \alpha\zeta_n \frac{H_n V_{n-2}}{U_n}.\end{aligned}$$

We have here two values for $\mathfrak{B}_{n+2}U_{n+2}$; reconciling them, we find

$$0 = \beta + \alpha\zeta_n H_{n-2} + \alpha(\epsilon_n - 1)H_n + \alpha\,\delta_n H_{n+2},$$

or,

$$\beta = -\alpha(p_0^2 - 1)H_n(p_0).$$

With regard to the first pair of terms, we suppose C_2 arbitrary, and B_2, D_2 separated into two parts, of which the former are the same as when the spheroid is perfect, while the latter are of the order α, and fall in regularly with the above.

Hence we get

$$\begin{aligned}\psi = {} & I_2(q)[Bp + CI_2(p) + DH_2(p)] \\ & - \alpha(p_0^2 - 1)H_n(p_0)\left\{I_2(q)\left[\frac{p}{U_2} + \frac{H_2(p)}{V_2} + \frac{H_4(p)}{U_4}\right]\right. \\ & \qquad + I_4(q)\left[\frac{H_2(p)}{U_4} + \frac{H_4(p)}{V_4} + \frac{H_6(p)}{U_6}\right] + \dots \\ & \qquad \left. + I_n(q)\left[\frac{H_{n-2}(p)}{U_n} + \frac{H_n(p)}{V_n}\right]\right\} \\ & - \alpha\left\{I_{n-2}(q)\left[\zeta_n \frac{H_n(p_0)}{U_n}H_{n-2}(p) - \zeta_n \frac{H_{n-2}(p_0)}{U_n}H_n(p)\right]\right. \\ & \quad + I_n(q)\left[-\frac{\zeta_n H_{n-2}(p_0)}{U_n}H_{n-2}(p)\right. \\ & \qquad \left. + \left(\overline{\epsilon_n - 1}\,\frac{H_{n+2}(p_0)}{U_{n+2}} - \frac{\zeta_n H_{n-2}(p_0)}{V_n}\right)H_n(p) - \frac{\delta_n H_{n+2}(p_0)}{U_{n+2}}H_{n+2}(p)\right] \\ & \quad \left. + I_{n+2}(q)\left[-\frac{\delta_n H_{n+2}(p_0)}{U_{n+2}}H_n(p) + \left(\delta_n \frac{H_{n+4}(p_0)}{U_{n+4}} - \delta_n \frac{H_{n+2}(p_0)}{V_{n+2}}\right)H_{n+2}(p)\right]\right\}.\end{aligned}$$

In this expression, B, C, D have the same values as the corresponding coefficients in (70), and

$$\alpha = \frac{\alpha_n}{2}\left(C + D\frac{dQ_1(p_0)}{dp_0}\right).$$

If n is odd we get a series exactly similar, except that the first terms are

$$-\alpha\left(p_0^2-1\right)\mathrm{H}_n(p_0)\left\{\mathrm{I}_3(q)\left[\frac{1}{\mathrm{U}_3}+\frac{\mathrm{H}_3(p)}{\mathrm{V}_3}+\frac{\mathrm{H}_5(p)}{\mathrm{U}_5}\right]+\&c.\right\}.$$

There is one exceptional case, viz., $n = 1$

$$p = p_0 + \alpha_1 q.$$

We have

$$\psi = \mathrm{I}_2(q)\left[\mathrm{B}p + \mathrm{CI}_2(p) + \mathrm{DH}_2(p)\right]$$
$$+\frac{\alpha}{\mathrm{Q}_2(p_0)}\left[\mathrm{H}_3(p)-\mathrm{H}_3(p_0)\right]\mathrm{I}_3(q),$$

where

$$\alpha = -\alpha_1\left[\mathrm{C}+\mathrm{D}\frac{d\mathrm{Q}_1(p_0)}{dp_0}\right]$$
$$\mathrm{C}+\mathrm{D}\frac{d\mathrm{Q}_1}{dp_0} = \frac{h^2\mathrm{V}p_0\left(p_0^2+3\right)}{\left(p_0^2-1\right)\mathrm{L}(p_0)}.$$

I shall now consider those two other functions, which, as we saw on p. 501, are solutions of $\mathrm{D}^2\psi = 0$, and make $\frac{d\psi}{dp} = \frac{d\psi}{dq} = 0$, when $p = p_0$.

One of them consists of a series of terms of odd order only, and the other of terms of even order only.

Consider the latter; put $\mathrm{C}_2 = \mathrm{B}_3 = 0$. Then we have the unlimited series of equations

$$\mathrm{B}_2p_0 + \mathrm{D}_2\mathrm{H}_2 + \mathfrak{B}_4\mathrm{H}_4 = 0 = \mathrm{B}_2 + \mathrm{D}_2\mathrm{Q}_1 + \mathfrak{B}_4\mathrm{Q}_3$$
$$\mathfrak{B}_4\mathrm{H}_2 + \mathrm{D}_4\mathrm{H}_4 + \mathfrak{B}_6\mathrm{H}_6 = 0 = \mathfrak{B}_4\mathrm{Q}_1 + \mathrm{D}_4\mathrm{Q}_3 + \mathfrak{B}_6\mathrm{Q}_5;$$
$$\cdots\cdots$$

whence, as on p. 506

$$\mathrm{D}_n\mathrm{V}_n = \mathfrak{B}_n\mathrm{U}_n = \mathfrak{B}_{n+2}\mathrm{U}_{n+2}$$
$$= \mathrm{B}_2\mathrm{U}_2.$$

And we find

$$\psi = \mathrm{B}_2\mathrm{U}_2\left\{\left[\frac{p}{\mathrm{U}_2}+\frac{\mathrm{H}_2(p)}{\mathrm{V}_2}+\frac{\mathrm{H}_4(p)}{\mathrm{U}_4}\right]\mathrm{I}_2(q)+\ldots\right.$$
$$\left.+\left[\frac{\mathrm{H}_{n-2}(p)}{\mathrm{U}_n}+\frac{\mathrm{H}_n(p)}{\mathrm{V}_n}+\frac{\mathrm{H}_{n+2}(p)}{\mathrm{U}_{n+2}}\right]\mathrm{I}_n(q)+\&c., \textit{ad inf.}\right\} \quad . \quad . \quad (76).$$

Similarly, the series consisting of terms of odd order is

$$\psi = \mathrm{B}_3\mathrm{U}_3\left\{\left[\frac{1}{\mathrm{U}_3}+\frac{\mathrm{H}_3(p)}{\mathrm{V}_3}+\frac{\mathrm{H}_5(p)}{\mathrm{U}_5}\right]\mathrm{I}_3(q)+\&c., \textit{ad inf.}\right\} \quad . \quad . \quad (76\mathrm{A}).$$

Now, the maximum value of the term

$$\frac{\mathrm{H}_n(p)}{\mathrm{U}_n} = \frac{\mathrm{H}_n(p)}{\mathrm{H}_n(p_0)\,\mathrm{Q}_{n-3}(p_0) - \mathrm{H}_{n-2}(p_0)\,\mathrm{Q}_{n-1}(p_0)}$$

occurs when $p = p_0$; and this maximum becomes very great when n is very great; for by p. 475, when n is great $\mathrm{H}_n(x) = -\frac{1}{2}\sqrt{\frac{\pi}{n^3}}\,\xi^{n-1}(1-\xi^2)^{\frac{1}{2}}$, and (HEINE, § 40) $\mathrm{Q}_n(x) = \sqrt{\frac{\pi}{n}}\,\xi^{n+1}(1-\xi^2)^{-\frac{1}{2}}$.

Hence

$$\frac{\mathrm{H}_n(p)}{\mathrm{U}_n} = -\frac{1}{2\sqrt{\pi}}\cdot\frac{\xi_0^{-n+3}(1-\xi_0^2)^{\frac{1}{2}}}{(n-3)^{-\frac{1}{2}} - n^{\frac{3}{2}}(n-2)^{-\frac{3}{2}}(n-1)^{-\frac{1}{2}}}$$

where $\xi_0 = p_0 - \sqrt{(p_0^2 - 1)}$, mod. ξ_0 being less than 1. Hence this expression becomes infinite with n; and no infinite series in which such a term is present can represent a motion within the limits of our discussion.

We could have foreseen that these series would be inapplicable, for we have already found one to which there is no objection, and it has been proved by HELMHOLTZ ('Wiss. Abh.,' vol. 1, XII.) that there is only one slow motion which satisfies the equations for a viscous liquid, and obeys definite boundary conditions.

3. *Hyperboloids.*

Let us now consider the motion which may take place within a hyperboloid of one sheet; say, of parameter q_0.

The conditions $d\psi/dp = d\psi/dq = 0$ are evidently satisfied by taking

$$\psi = \mathrm{D}_1 q + \mathrm{B}_3\mathrm{I}_3(q)$$

where

$$\mathrm{D}_1 + \mathrm{B}_3\mathrm{P}_2(q_0) = 0\,,$$

or

$$\psi = \frac{\mathrm{B}_3}{2}\,q\,(q^2 - 3q_0^2)$$

and

$$\mathrm{B}_3 = -2\mathrm{V}h^2/3\,(1 - q_0^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (77)$$

That is to say, the stream surfaces are hyperboloids of the system confocal with the boundary.

The velocities which (77) gives are

$$\left.\begin{aligned} \mathrm{U} &= -\frac{\mathrm{V}}{1-q_0^2}\,\frac{(q^2-q_0^2)\,p}{(p^2-q^2)\,i}\sqrt{\frac{1-q^2}{1-p^2}} \\ \mathrm{W} &= \frac{\mathrm{V}}{1-q_0^2}\,\frac{(q^2-q_0^2)}{p^2-q^2}\,q \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (77\mathrm{A})$$

where V is the velocity at the centre from right to left.

In the particular case of a flat wall pierced by a circular hole, we have $q_0 = 0$, and

$$\left.\begin{aligned} \mathrm{U} &= -\mathrm{V}\frac{pq^2\sqrt{(1-q^2)}}{i\,(p^2-q^2)\sqrt{(1-p^2)}} \\ \mathrm{W} &= \mathrm{V}\frac{q^3}{p^2-q^2} \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (77\mathrm{B})$$

both of which vanish at the edge.

4. *Slipping Motions at the Boundary.*

Approaching now as near as may be possible to a discussion of the solutions when the liquid slips at the surface of the obstacle, we find, without difficult or very long calculation, that if on p. 495 we put $\alpha, \beta, \gamma, = p, q, \phi$, and consequently $u, v, = \mathrm{P}, \mathrm{Q}$ of p. 478, while $w = 0$, we find, I say, $a = b = 0$,

$$2c = \frac{1}{(p^2-q^2)\sqrt{(p^2-1\,.\,1-q^2)}}\left\{(1-q^2)\frac{d^2\psi}{dq^2} - (p^2-1)\frac{d^2\psi}{dp^2} + \frac{2q\,(1-q^2)}{p^2-q^2}\frac{d\psi}{dq} + \frac{2p\,(p^2-1)}{p^2-q^2}\frac{d\psi}{dp}\right\} \quad \ldots\ldots \quad (78)$$

Now if T is the tangential stress per unit area at any point of the obstacle, whether spheroid or hyperboloid, $\mathrm{T} = 2\mu c$, and a boundary condition being that this tangential stress shall be proportional to the velocity of the liquid, at the surface of the obstacle, we have

for spheroid (p_0) $\quad \mathrm{T} = \beta\mathrm{Q}, \quad \mathrm{P} = 0$, when $p = p_0$
for hyperboloid (q_0) $\quad \mathrm{T} = \beta\mathrm{P}, \quad \mathrm{Q} = 0$, when $q = q_0$

where β is a constant, becoming very large when the slipping is very slight.

Now

$$\begin{aligned} \psi = \mathrm{C}_0 + \mathrm{D}_1 q &+ \mathrm{I}_2(q)\,[\mathrm{B}_2 p + \mathrm{C}_2\mathrm{I}_2(p) + \mathrm{D}_2\mathrm{H}_2(p) + \mathfrak{B}_4\mathrm{H}_4(p)] \\ &+ \mathrm{I}_3(q)\,[\mathrm{B}_3 + \mathrm{D}_3\mathrm{H}_3(p) + \mathfrak{B}_5\mathrm{H}_5(p)] \\ &+ \mathrm{I}_4(q)\,[\mathfrak{B}_4\mathrm{H}_2(p) + \mathrm{D}_4\mathrm{H}_4(p) + \mathfrak{B}_6\mathrm{H}_6(p)] + \ldots \end{aligned} \quad (69)$$

and remembering

$$\frac{d}{dx}\cdot\mathrm{I}_n(x) = \mathrm{P}_{n-1}(x)$$

$$\frac{d^2}{dx^2}\cdot\frac{\mathrm{I}_n(x)}{\mathrm{H}_n(x)} = -\frac{n\,(n-1)}{1-x^2}\cdot\frac{\mathrm{I}_n(x)}{\mathrm{H}_n(x)},$$

we see that if P, and, therefore, $d\psi/dq = 0$, for definite values of p, we also have $d^2\psi/dq^2 = 0$.

Hence the above conditions become, *for spheroidal obstacle*, $p = p_0$,

$$\frac{d\psi}{dq} = 0.$$

$$\frac{d^2\psi}{dp^2} - \frac{2p_0}{p_0^2 - q^2}\frac{d\psi}{dp} = \frac{\beta}{\mu}\sqrt{\frac{p_0^2 - q^2}{p_0^2 - 1}}\frac{d\psi}{dp}$$

when

$$p = p_0\,; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (79)$$

for hyperboloid, $q = q_0$,

$$\frac{d^2\psi}{dq^2} - \frac{p^2 - 1}{1 - q_0^2}\frac{d^2\psi}{dp^2} + \frac{2q_0}{p^2 - q_0^2}\frac{d\psi}{dq} = \frac{\beta}{\mu}\sqrt{\frac{p^2 - q_0^2}{1 - q_0^2}}\frac{d\psi}{dq} \quad . \quad . \quad . \quad . \quad (79\text{A})$$

From these equations we can detect at once a solution in a particular case, viz., the flux through a circular hole in a plane wall, when the result we have already obtained on p. 509, viz., $\psi = - \mathrm{V}a^2/3 \,.\, q^3$ (a being the radius of the hole, and V the velocity at its centre) still applies. In fact, the tangential stress at any point of the wall is zero.

But it is obvious that it is out of the question to apply, in general, the direct method of assuming (69) and substituting, in order to find functions which satisfy (79) and (79A), and it is also clear that the solution for the spheroid is not, as in the case of the sphere, of like simplicity with the case when the slipping is zero. In one case I have followed out the process of solution, viz., when the obstacle is a circular disc, and the slipping slight. We then have $p_0 = 0$, and

$$q\frac{d\psi}{dp} = \frac{\mu}{\beta}\frac{d^2\psi}{dq^2},$$

and assuming that the solution does not differ greatly from the case when there is no slipping, I at first neglected, on the right, all terms except those entering in (70), p. 502.

This led to a set of equations for finding the coefficients

$$\mathfrak{B}_5\mathrm{U}_5 - \mathfrak{B}_3\mathrm{U}_3 = -\,5\,.\,\tfrac{2}{3}\,.\,\frac{2\mu a^2\mathrm{V}}{\beta}$$

$$\mathfrak{B}_7\mathrm{U}_7 - \mathfrak{B}_5\mathrm{U}_5 = -\,\tfrac{9\,.\,4}{5}\,.\,(\tfrac{2}{3})^2\,.\,\frac{2\mu a^2\mathrm{V}}{\beta}$$

. . . .

$$\mathfrak{B}_{2n+1}\mathrm{U}_{2n+1} - \mathfrak{B}_{2n-1}\mathrm{U}_{2n-1} = -\,\frac{(4n-3)\,(2n-2)}{2n-1}\left(\frac{2n+4\ldots 2}{2n-3\ldots 3}\right)^2 . \,\frac{2\mu a^2\mathrm{V}}{\beta}.$$

Now, when n is very great, the limit of

$$\frac{2\,.\,4\ldots 2n}{3\,.\,5\ldots 2n+1} \quad \text{is} \quad \tfrac{1}{2}\sqrt{\frac{\pi}{n}}\,.$$

Hence, when n is great,

$$\mathfrak{B}_{2n+1}\mathrm{U}_{2n+1} - \mathfrak{B}_{2n-1}\mathrm{U}_{2n-1} = -\,\frac{2\pi\mu a^2\mathrm{V}}{\beta}$$

That is, the coefficients $\mathfrak{B}$ become infinite when n is infinite, and the series which the expression (69) would require us to use for ψ is divergent, showing that the assumption that ψ may be written in a series proceeding like (69) is, in this case, not justified.

It will be seen from the last section, and from p. 508, that an infinite series of the form, satisfying common boundary conditions, frequently becomes divergent. This would not always seem to show that there is no corresponding possible motion, but rather that the mode of expression chosen is not permissible.

Now it is easy to see that there is no solution of $D^2\psi = 0$, consisting of a finite number of terms of (69) which obeys (79) or (79A), the solution $\psi = q^3$ excepted. Hence it seems probable that a series of the form (69) is not appropriate for expressing the motion of a viscous liquid, slipping at the surface of a spheroid or hyperboloid.*

NOTE. (Added, June, 1891.)

At the suggestion of the referees, I add a more detailed consideration of the motion past a disc, and through a hole in a wall.

A circular disc, of radius a, is moved perpendicular to its plane with velocity V; the stream-lines relative to it are given by

$$\psi = -2a^2 V I_2(q)\left[I_2(p) + \frac{2i}{\pi}\{p + H_2(p)\}\right],$$

or, writing $p = i\mathfrak{p}$,

$$= -\frac{a^2 V}{\pi}(1 - q^2)[(1 + \mathfrak{p}^2)\tan^{-1}\mathfrak{p} - \mathfrak{p}].$$

The stresses at any point of the liquid are

$$P = -p + 2\mu e;\quad Q = -p + 2\mu f;\quad R = -p + 2\mu g;\quad S = 0 = T;\quad U = 2\mu c,$$

and p being the mean of the pressures normal to any three mutually orthogonal surfaces, I shall hereafter allude to it as the "mean pressure" at the point.

Let us first find p.

If $Q = -\frac{p}{\rho} - V$, we have†

$$-dQ = \frac{\nu}{\varpi}\left(\frac{d\chi}{d\varpi}dz - \frac{d\chi}{dz}d\varpi\right)$$

where $\chi = D\psi$.

Transforming to the elliptic coordinates p, q,

$$-dQ = -\frac{\nu}{h}\left[\frac{1}{1-q^2}\frac{d\chi}{dp}dq + \frac{1}{1-p^2}\frac{d\chi}{dq}dp\right]$$

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[*] See p. 515.

[†] BASSET, vol. 2, p. 262.

which is an exact differential by (3e) p. 494; and gives, in the case of any spheroid,

$$Q = \frac{\nu V}{h L(p_0)} \cdot \frac{q}{p^2 - q^2}$$

or in the case of the disc, of radius a,

$$Q = \frac{4\nu V}{\pi a} \cdot \frac{q}{p^2 - q^2},$$

and since there are no applied forces, the variable part of the mean pressure is

$$p = \frac{4V\mu}{\pi a} \cdot \frac{q}{\mathfrak{p}^2 + q^2}.$$

Again, by the formulæ, p. 496,

$$e = \frac{1}{h}\sqrt{\frac{p^2-1}{p^2-q^2}} \cdot \frac{du}{dp} - \frac{v}{h} \cdot \frac{q\sqrt{(1-q^2)}}{(p^2-q^2)^{\frac{3}{2}}};$$

$$f = \frac{1}{h}\sqrt{\frac{1-q^2}{p^2-q^2}} \cdot \frac{dv}{dq} + \frac{u}{h} \cdot \frac{p\sqrt{(p^2-1)}}{(p^2-q^2)^{\frac{3}{2}}};$$

$$g = \frac{up}{\sqrt{(p^2-1 \,.\, p^2-q^2)}} - \frac{vq}{\sqrt{(1-q^2 \,.\, p^2-q^2)}};$$

$$a = b = 0;$$

$$2c = \frac{1}{h}\sqrt{\frac{1-q^2}{p^2-q^2}} \cdot \frac{du}{dq} + \frac{1}{h}\sqrt{\frac{p^2-1}{p^2-q^2}} \cdot \frac{dv}{dp} - \frac{vp\sqrt{(p^2-1)}}{h(p^2-q^2)^{\frac{3}{2}}} + \frac{uq\sqrt{(1-q^2)}}{h(p^2-q^2)^{\frac{3}{2}}};$$

whence, in the case of the disc,

$$e = \frac{2V}{a\pi} \cdot \frac{p^2 q(1+q^2-2p^2)}{(p^2-q^2)^2(p^2-1)} = \frac{2V}{a\pi} \cdot \frac{\mathfrak{p}^2 q(1+q^2+2\mathfrak{p}^2)}{(\mathfrak{p}^2+q^2)^2(\mathfrak{p}^2+1)};$$

$$f = \frac{2V}{a\pi} \cdot \frac{p^2 q}{(p^2-q^2)^2} = -\frac{2V}{a\pi} \cdot \frac{\mathfrak{p}^2 q}{(\mathfrak{p}^2+q^2)^2};$$

$$g = \frac{2V}{a\pi} \cdot \frac{p^2 q}{(p^2-q^2)(p^2-1)} = -\frac{2V}{a\pi} \cdot \frac{\mathfrak{p}^2 q}{(\mathfrak{p}^2+q^2)(\mathfrak{p}^2+1)};$$

$$c = \frac{2V}{a\pi} \cdot \frac{pq^2}{(p^2-q^2)^2}\sqrt{\frac{1-q^2}{p^2-1}} = \frac{2V}{a\pi} \cdot \frac{\mathfrak{p}q^2}{(\mathfrak{p}^2+q^2)^2}\sqrt{\frac{1-q^2}{\mathfrak{p}^2+1}}.$$

These all vanish on the surface of the disc ($p = 0$). Hence the total pressure on the disc is got by integrating the mean pressure over both sides of its surface. This gives $16V\mu a$, agreeing with the known value for any ellipsoid.*

I append tracings of the curves, const. $= \psi/a^3$, $p/2\mu$, e, f, g, c, with numerical values calculated on the supposition that $V/\pi a = 1$. The first shows the stream-lines;

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* BASSET, § 497.

the second, the mean pressure at any point; the third, fourth, and fifth, the departure from this mean in directions respectively normal to the ellipsoids, to the hyperboloids, and to planes through the axis; the last shows the tangential stress for these particular directions of normal pressure.

With the exception of g, which has a maximum at the point on the axis distant one radius from the disc, all increase if we approach the edge of the disc in particular directions, and we can make this increase very large; but at the edge itself they appear to be indeterminate, not infinite, all the surfaces of equal stress passing through one and the same line; and there seems no ground to suppose that it is impossible to impress a sufficiently great external pressure upon the whole liquid to prevent the pressures becoming negative. But it suggests that the test of zero or finite velocity at a sharp edge is not sufficient to ensure the motion being admissible.

Next, consider motion through a circular hole in a wall.

We have, for any hyperboloid,

$$\psi = \frac{2\mathrm{V}h^2}{3(1-q_0^2)}(q^3 - 3qq_0^2)$$

where V is the velocity at the centre.

Hence

$$\mathrm{Q} = -\frac{4\nu\mathrm{V}}{h(1-q_0^2)}\left\{\frac{p}{p^2-q^2} - \tfrac{1}{2}\log\frac{1+p}{1-p}\right\}$$

and the mean pressure is

$$p = \frac{4\mu\mathrm{V}}{ih(1-q_0^2)}\left[\tan^{-1}\mathfrak{p} + \frac{\mathfrak{p}}{\mathfrak{p}^2+q^2}\right],$$

and for a flat wall ($q_0 = 0$)

$$p = \frac{4\mu\mathrm{V}}{a}\left[\tan^{-1}\mathfrak{p} + \frac{\mathfrak{p}}{\mathfrak{p}^2+q^2}\right],$$

where a is the radius of the hole.

The difference between the pressures at infinity is

$$\Pi = \frac{4\mu\mathrm{V}\pi}{ih(1-q_0^2)}.$$

Again, if F is the total flux per unit time through the aperture,

$$\mathrm{F} = \frac{\mathrm{V}\pi h^2}{3(1-q_0^2)}(q_0-1)^2(2q_0+1).$$

Hence, a being the radius of the aperture, so that $a = ih\sqrt{(1-q_0^2)}$

$$\mathrm{F} = \frac{a^3}{12\mu}\cdot\frac{(2q_0+1)(1-q_0)^{\frac{1}{2}}}{(1+q_0)^{\frac{3}{2}}}\cdot\Pi,$$

or, in the case of the plane wall,

$$\mathrm{F} = \frac{a^3}{12\mu}\,\Pi.$$

By applying the formulæ given at the beginning of this note

$$e = \frac{2\mathrm{V}i}{a} \cdot \frac{pq^2(1 + q^2 - 2p^2)}{(p^2 - q^2)^2(p^2 - 1)} = \frac{2\mathrm{V}}{a} \cdot \frac{\mathfrak{p}q^2(1 + q^2 + 2\mathfrak{p}^2)}{(\mathfrak{p}^2 + q^2)^2(\mathfrak{p}^2 + 1)};$$

$$f = \frac{2\mathrm{V}i}{a} \cdot \frac{pq^2}{(p^2 - q^2)^2} = \frac{2\mathrm{V}}{a} \cdot \frac{\mathfrak{p}q^2}{(\mathfrak{p}^2 + q^2)^2};$$

$$g = \frac{2\mathrm{V}i}{a} \cdot \frac{pq^2}{(p^2 - q^2)(p^2 - 1)} = -\frac{2\mathrm{V}}{a} \cdot \frac{\mathfrak{p}q^2}{(\mathfrak{p}^2 + q^2)(\mathfrak{p}^2 + 1)};$$

$$c = \frac{2\mathrm{V}i}{a} \cdot \frac{p^2q}{(p^2 - q^2)^2} \sqrt{\frac{1 - q^2}{p^2 - 1}} = \frac{2\mathrm{V}}{a} \cdot \frac{\mathfrak{p}^2q}{(\mathfrak{p}^2 + q^2)^2} \sqrt{\frac{1 - q^2}{\mathfrak{p}^2 + 1}}.$$

Tracings of sections of these surfaces are given with numerical values on the supposition that $\mathrm{V}/a = 1$, as also for $p/2\mu$. With the exception of the mean pressure, the curves are in general character the same as for motion past a disc.

The surfaces of mean pressure are the same, except for the numerical values attached to them, for all planetary spheroids and for all hyperboloidal boundaries.

Since the reading of this paper, I have found a proof* that the expansion of any function of q in a linear series of $\mathrm{I}(q)$'s is convergent where q is less than unity. This throws more light on the problem of slipping motion past a disc, attacked in Chapter VI., for it shows that if the expression for any function ψ is a divergent series, then there is no corresponding finite motion.

The work there was conducted on the assumption that the motion, if existing, did not differ greatly when the slipping was small from the case when there was none; and, therefore, we have not a rigorous proof that there is no such motion, though this seems to make it highly probable. As it is not easy to see how there should be a motion with slipping for one particular case of spheroid, viz., a sphere, and none for another, viz., a disc, unless the former is a wholly exceptional case, it is interesting to notice that the probability of it being exceptional is borne out by the analysis (p. 511), for if we approach the case of the sphere, by putting $p = r/h$ where h is indefinitely small in the equation of condition (79), q ultimately disappears from the equation.

It must be remembered that the search for steady motions in viscous liquids proceeds on an assumption, viz., that if the motion is started from rest, the limit of the rate of change of velocity is zero at all points when the time is infinite; therefore, such motions must always be exceptional, and, again, the analysis bears us out in expecting this, for the same thing will be noticed in the two solutions which I give that STOKES pointed out in the case of the sphere—that the possibility of satisfying the boundary conditions $\frac{d\psi}{dq} = 0 = \frac{d\psi}{dp}$ depends on the presence of an irregular term in the series (p. 501), namely, $\mathrm{B}_2p\mathrm{I}_2(q)$ for spheroids, and $\mathrm{B}_3\mathrm{I}_3(q)$ for hyperboloids.

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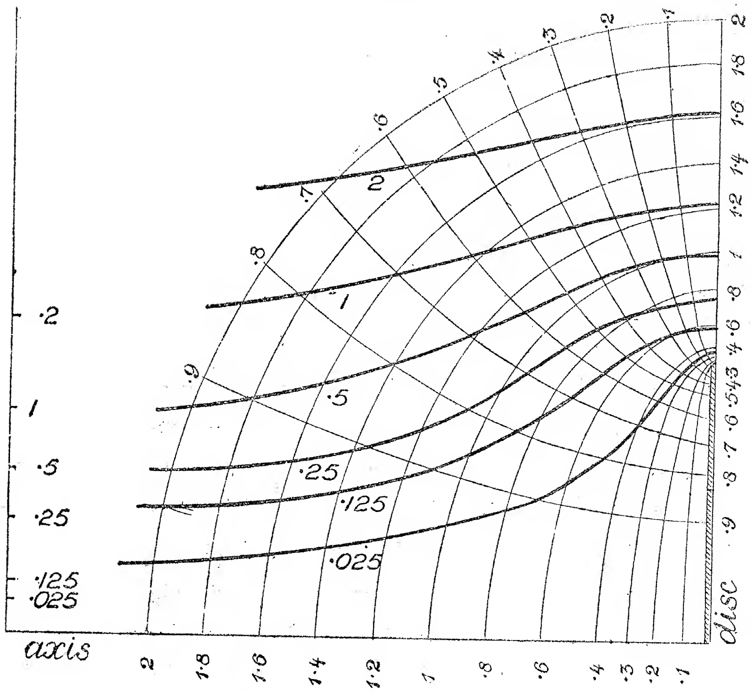
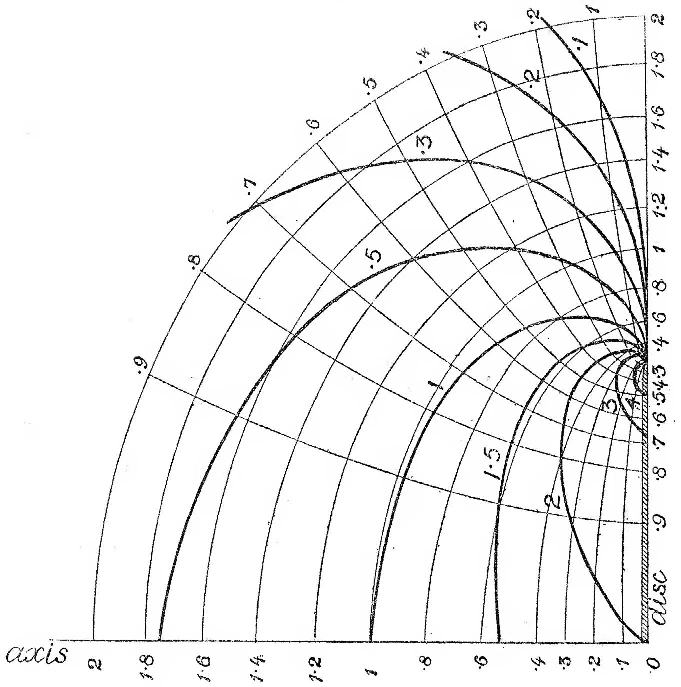
* Inserted in Chapter II.

Fig. 1.



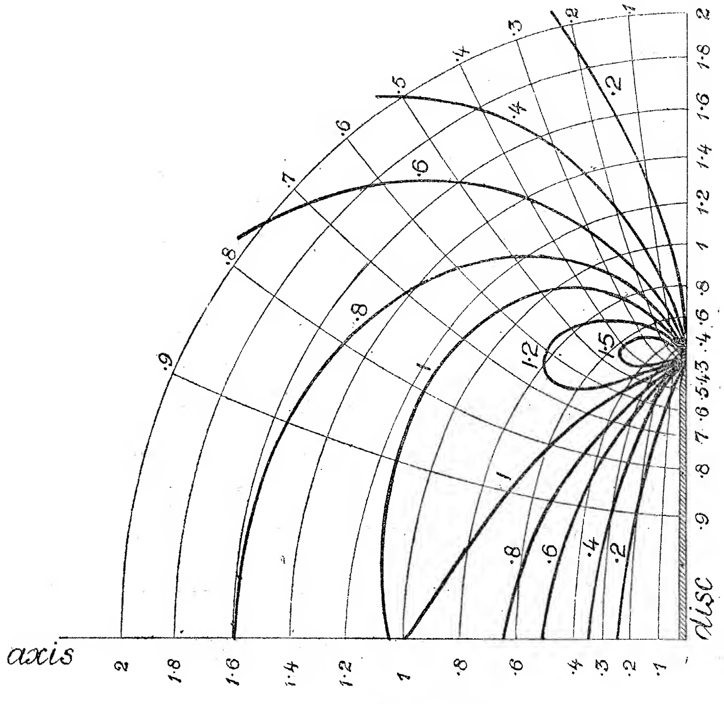
$\psi \cdot \frac{\pi}{Va^2}$ for disc.

Fig. 2.



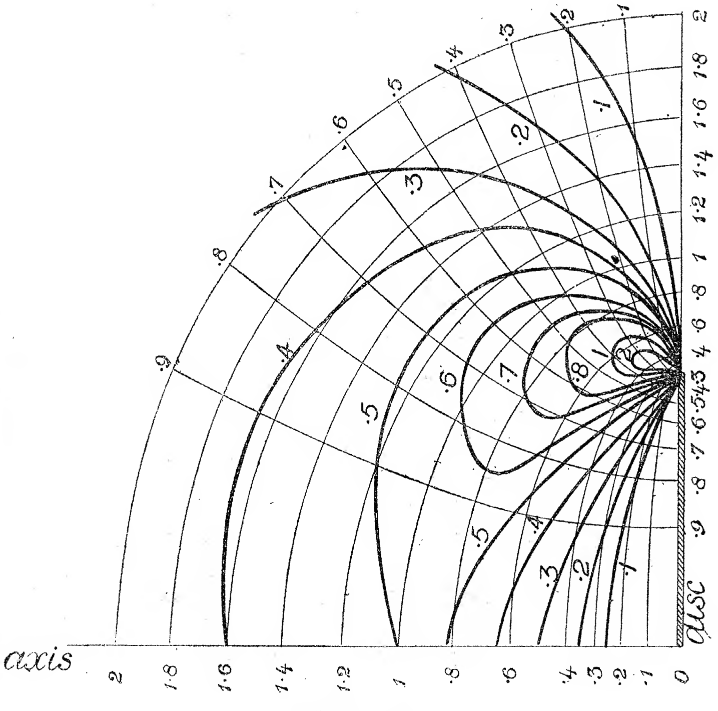
$\frac{p}{2\mu} \cdot \frac{a\pi}{V}$ for disc.

Fig. 3.



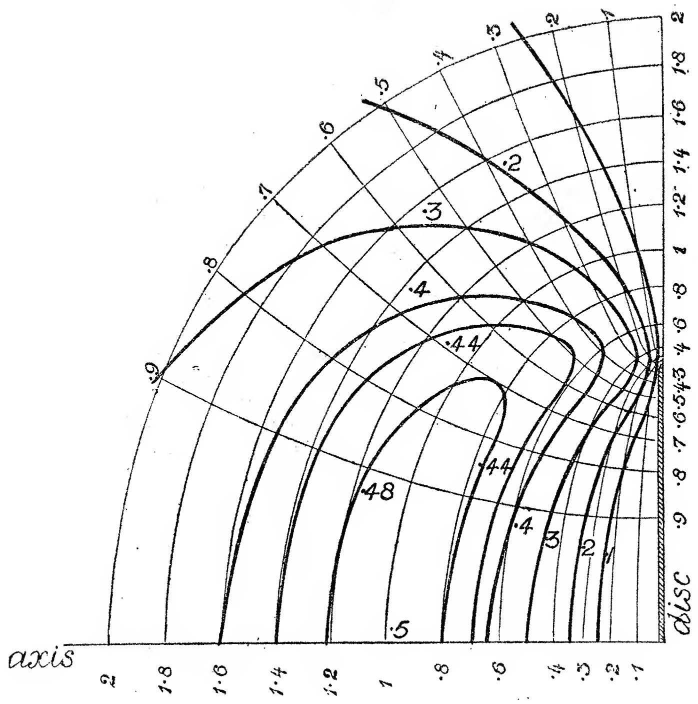
$e \cdot \frac{a\pi}{V}$ for disc.

Fig. 4.



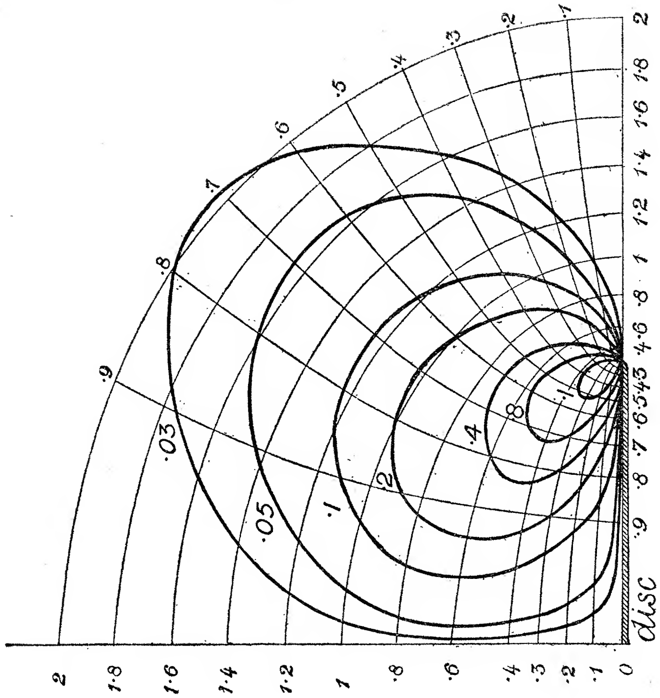
$f \cdot \frac{a\pi}{V}$ for disc.

Fig. 5.



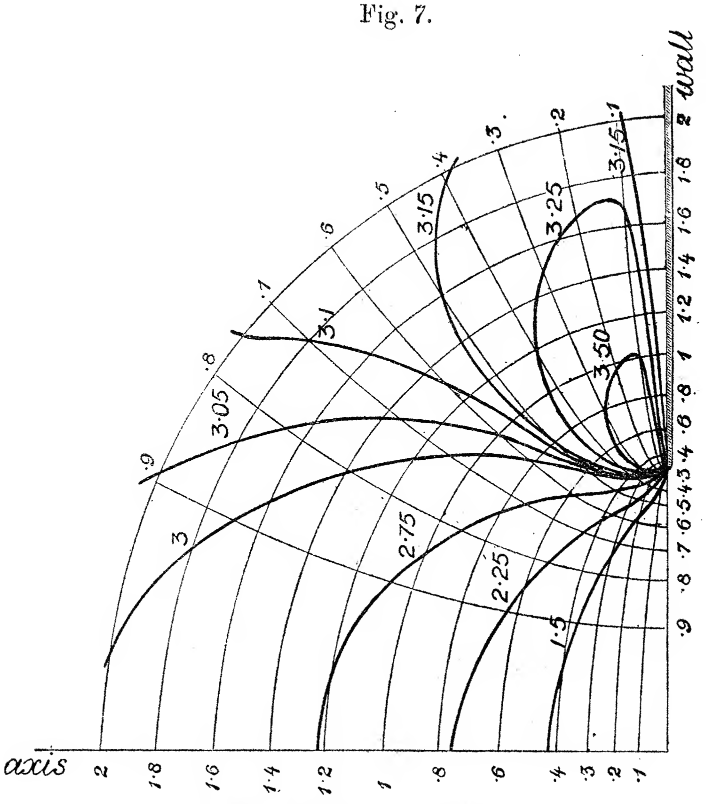
$g \cdot \frac{a\pi}{V}$ for disc.

Fig. 6.



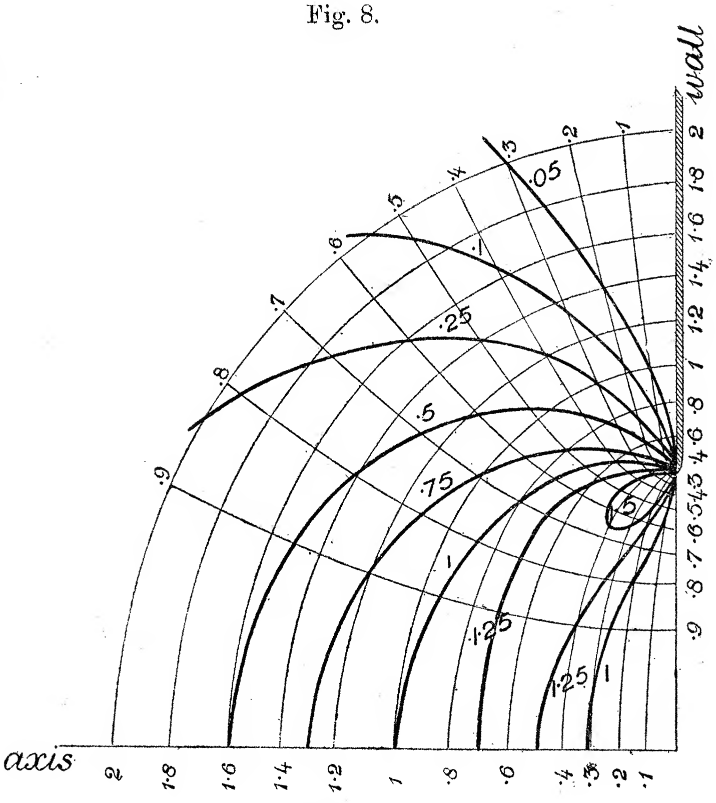
$c \cdot \frac{a\pi}{V}$ for disc.

Fig. 7.



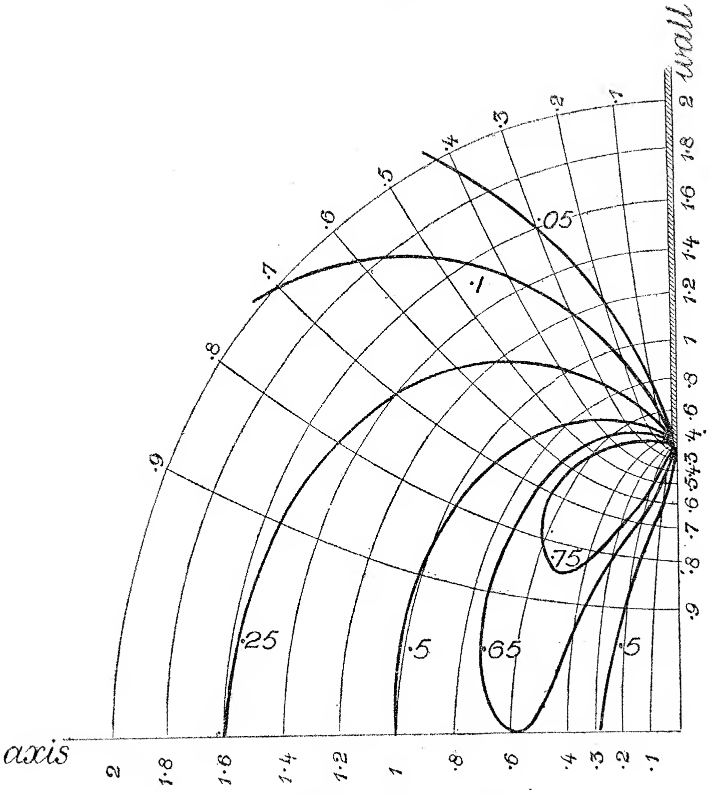
$\frac{p}{2\mu} \cdot \frac{a}{V}$ for hole in wall.

Fig. 8.



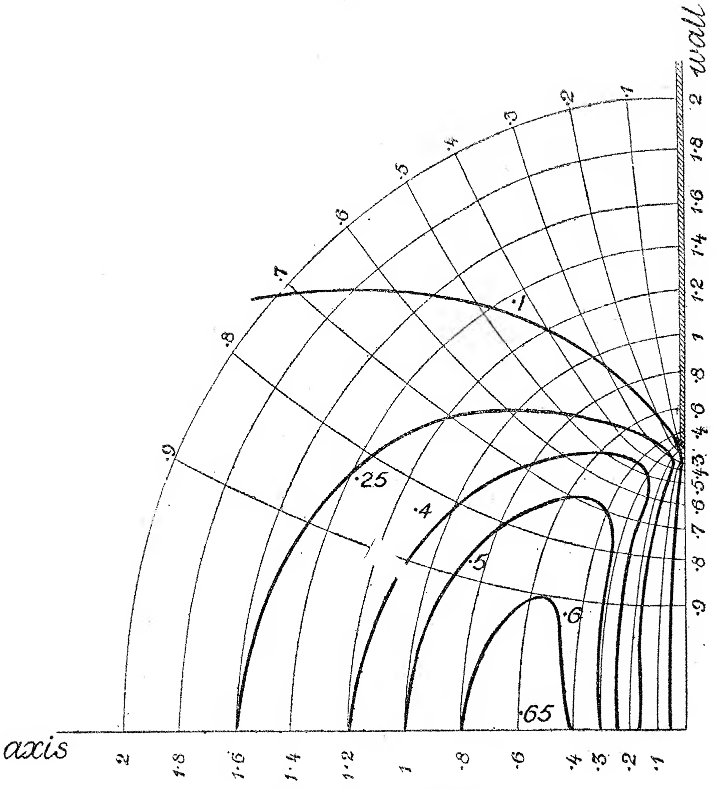
$e \cdot \frac{a}{V}$ for hole in wall.

Fig. 9.



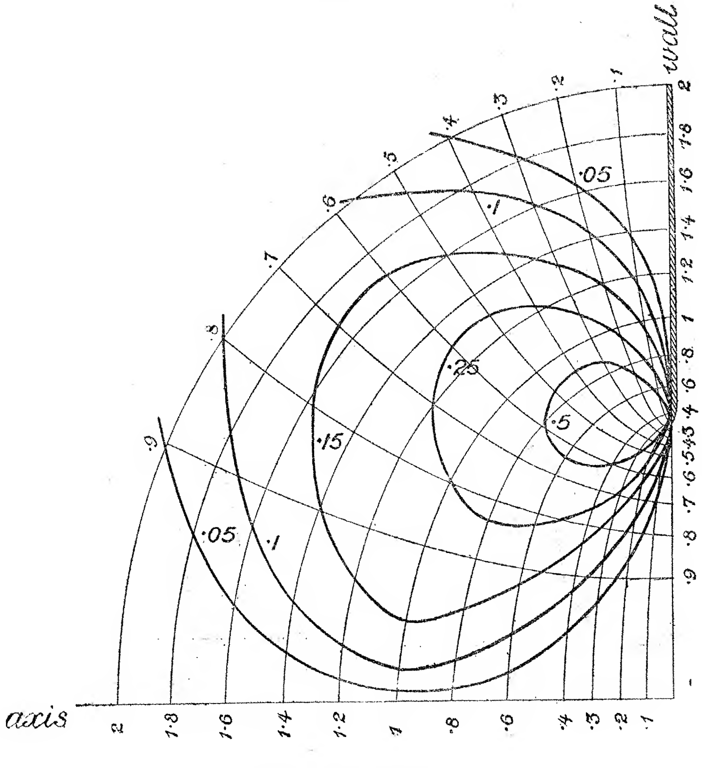
$f \cdot \frac{a}{V}$ for hole in wall.

Fig. 10.



$g \cdot \frac{a}{V}$ for hole in wall.

Fig. 11.



$c \cdot \frac{a}{V}$ for hole in wall.